Scale Invariant Summary Functions for Stationary and Inhomogeneous Point Processes

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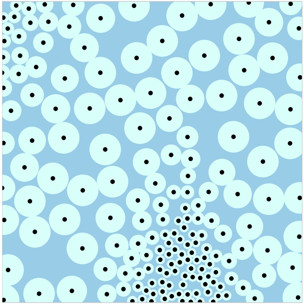
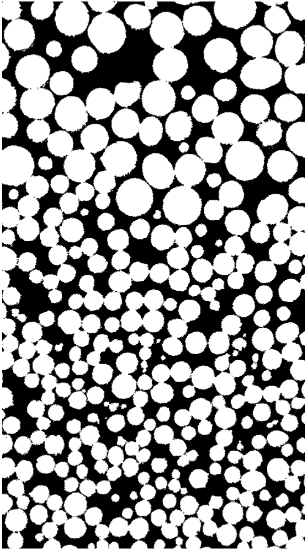




Section of a gradient sinter filter made of bronze powder

- Locally scaled point processes
- Scale invariant versions of summary functions
- Locally scaled summary functions
- Analysis of the bronze data set

A point process X on $\mathcal{X} \subseteq \quad ^d$ is a locally scaled version of a stationary point process X_0 with respect to the scale function $c : \quad ^d \rightarrow (0, \infty)$ if to every $B \in \mathcal{B}^d$ there exists $\tilde{B} \supseteq B$ such that

$$c(x) \equiv \tilde{c} \text{ for all } x \in \tilde{B}$$

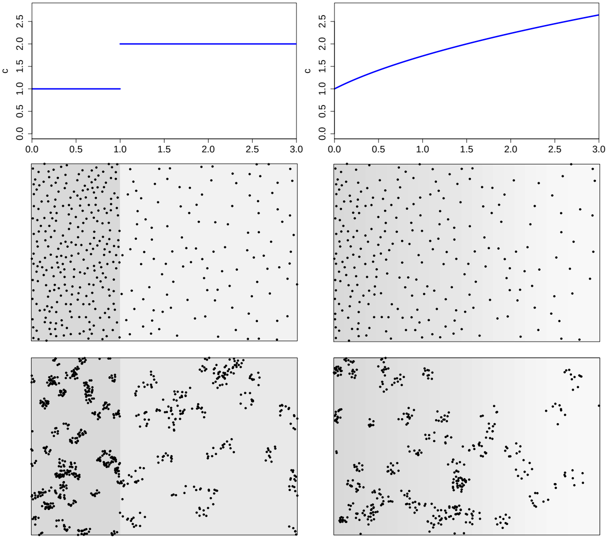
implies

$$X \cap B \stackrel{\mathrm{d}}{=} \tilde{c} X_0 \cap B.$$

Consequences:

$$\lambda_X(x) = \tilde{c}^{-d} \lambda_{X_0} \quad \text{for } x \in B,$$

$$\mathbf{E}_x^!(X(b(x,r))) = \mathbf{E}_x^!(X_0(b(x, r/\tilde{c}))) \quad \text{if } b(x,r) \subset B.$$



scale function

Matérn hard core point process

Matérn cluster point process

T: any statistics of a stationary point process X with intensity λ.
T^*: corresponding scale invariant version,

$$T^* := T \circ S_{\sqrt[d]{\lambda}},$$

S_c: scaling operator, $S_c x = cx$, $x \in {}^d$.

Example: K-function

$$\begin{aligned} K_X^*(r) &= K_{X^*}(r) \\ &= \frac{1}{\lambda_{X^*}} \mathbf{E}_o^! \big(X^*(b(o,r))\big) = \mathbf{E}_o^! \big(X(b(o, r/\sqrt[d]{\lambda}))\big) = \lambda K_X(r/\sqrt[d]{\lambda}) \end{aligned}$$

with $X^* = S_{\sqrt[d]{\lambda}} X = \sqrt[d]{\lambda} X$: point process X rescaled to unit intensity.

K-function

$$K^*(r) = \mathbf{E}_x^!\big(X(b(x, r/\sqrt[d]{\lambda}))\big) = \lambda K(t/\sqrt[d]{\lambda})$$

pair correlation function

$$g^*(r) = \frac{\mathrm{d}K^*(r)}{\mathrm{d}r} \Big/ d\omega_d r^{d-1} = g(t/\sqrt[d]{\lambda}) \quad (\omega_d\text{: unit sphere volume})$$

L-function

$$L^*(r) = \sqrt[d]{K^*(r)/\omega_d} = \sqrt[d]{\lambda} L(t/\sqrt[d]{\lambda})$$

nearest neighbour distance distribution, aka G-function

$$D^*(r) = \mathbf{P}_x^!\big(X(b(x, r/\sqrt[d]{\lambda})) > 0\big) = D(t/\sqrt[d]{\lambda})$$

spherical contact distribution, aka F-function

$$H_s^*(t) = \mathbf{P}\big(X(b(o, r/\sqrt[d]{\lambda})) > 0\big) = H_s(t/\sqrt[d]{\lambda})$$

Example: K^*-function

$$K^*(r) = \mathbf{E}^!_x\big(X(b(x, r/\sqrt[d]{\lambda}))\big) = \mathbf{E}^!_x\big(X(b_\lambda(x, r))\big)$$

with

$$b_\lambda(x, r) := \{y \in \quad {}^d : d_\lambda(x, y) \le r\},$$
$$d_\lambda(x, y) := \int_{[x,y]} \sqrt[d]{\lambda(u)}\,\mathrm{d}u,$$

$[x, y]$: line segment connecting x and y.

Individual K^*-function

$$K^*_x(r) = \mathbf{E}^!_x\big(X(b_\lambda(x, r))\big),$$

averaged over a window W:

$$K^*_W(r) = \frac{1}{\Lambda(W)} \int_W K^*_x(r)\Lambda(\mathrm{d}x).$$

Proposition: Let X be a locally scaled version of the unit rate point process point process X_0 with respect to the scale function c.
Assume that

$$c(u) \equiv \tilde{c} \text{ for all } u \in \tilde{B} \supset b(x, \tilde{c}r),$$

with $\tilde{B}$ large enough to ensure that

$$X \cap b(x, \tilde{c}r) \stackrel{\mathrm{d}}{=} \tilde{c}X_0 \cap b(x, \tilde{c}r).$$

Then

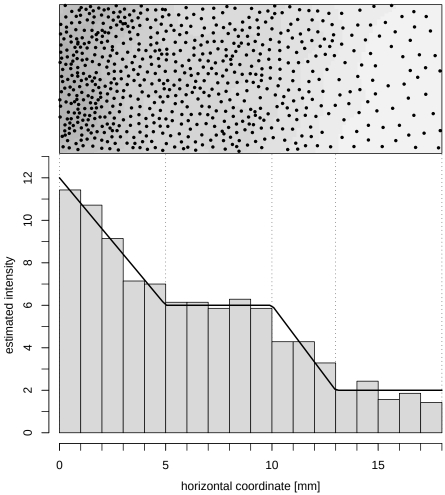
$$K_x^*(r) = K_0(r).$$

If the scale function does not vary very much within the scaled distance r,

$$K_W^*(r) \approx K_0(r)$$

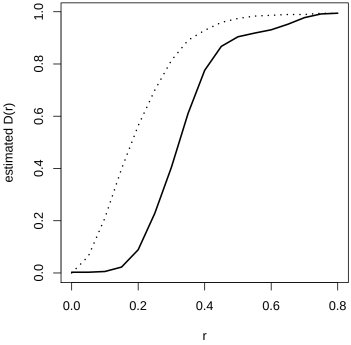
very closely.

Estimated intensity

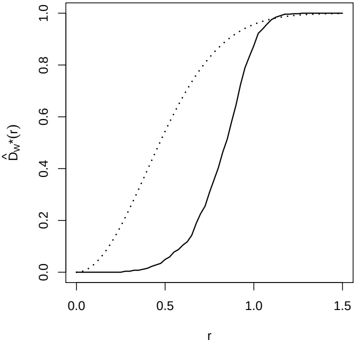


Nearest neighbour distance distribution

D (non scaled)

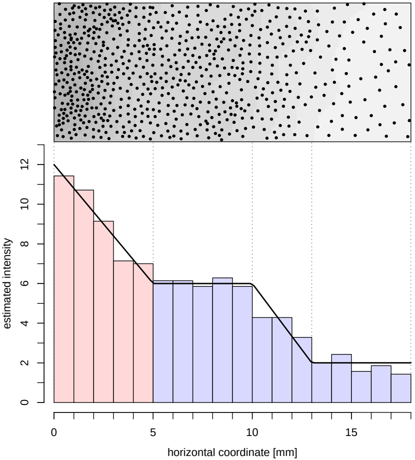


D^* (locally scaled)

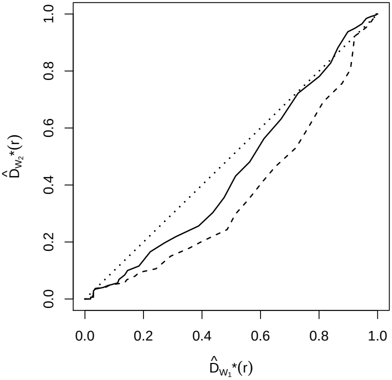


Dotted lines: corresponding Poisson point process

Division into subwindows

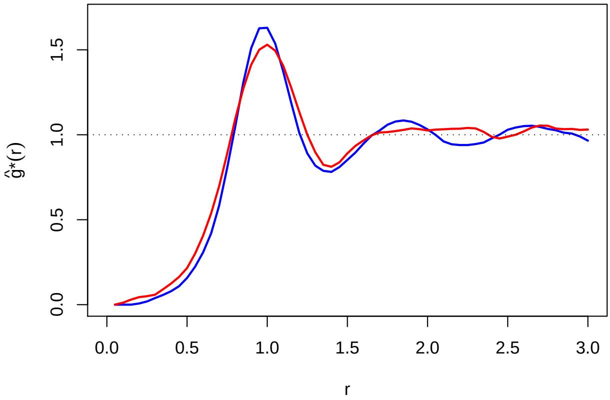


PP-plot of the D^* for the two subwindows

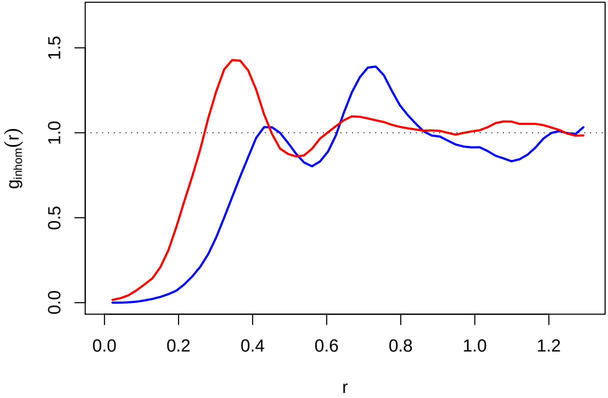


Dashed lines: horizontal division of the window into two halves

Locally scaled pair correlation function g^* for the two subwindows



Inhomogeneous pair correlation function g_{inhom} for the two subwindows



Hahn, U., Jensen, E. B. V., van Lieshout, M. N. M., and Nielsen, L. S. (2003): Inhomogeneous spatial point processes by location-dependent scaling.
Advances in Applied Probability (SGSA), 35:319–336.

Prokešová, M., Hahn, U., and Jensen, E. B. V. (2006): Statistics for locally scaled point processes.
In Baddeley, A., Gregori, P., Mateu, J., Stoica, R., and Stoyan, D. (editors), *Case Studies in Spatial Point Process Modelling*, volume 185 of *Lecture Notes in Statistics*, pages 99–123. Springer, New York.