ABSOLUTE φ-SUMMABILITY FACTORS WITH A POWER FOR A^{α}-METHODS *

Simson Baron and Rüdiger Kiesel

Received:

ABSTRACT. We give necessary and sufficient conditions for a sequence (ε_n) of complex numbers to be summability factors of type $(\varphi\text{–}|A|_r, \varphi\text{–}|B|_s)$ for various (r,s) combinations, where A is an A^{α} matrix of summability and B is a triangular matrix of summability satisfying certain regularity conditions. As examples we discuss summability factors of the above type in case $B = Z_{\beta}$, the method of Silverman-Szasz, and in case $A = M_a$ and $B = M_b$ with M_a and M_b being summability methods of weighted means.

AMS subject classification (1991): 40C05, 40D15, 40F05

1 Introduction and main result

Let $\varphi = (\varphi_n)$ be a sequence of positive numbers and

$$\sum u_n \tag{1}$$

be a series of complex numbers u_n. Unless otherwise indicated free indices always run through all values $0, 1, 2, \ldots$, and summation is likewise over $0, 1, 2, \ldots$. Let A be a triangular method of summability with the matrix (α_{nk}) of series-to-sequence transformation and with the matrix $(\bar{\alpha}_{nk})$ of series-to-series transformation. We set

$$v_n = \sum_{k=0}^{n} \bar{\alpha}_{nk} u_k. \tag{2}$$

*This research was enabled by the generous support of the Minerva Foundation in Germany at the Emmy-Noether-Institute at Bar-Ilan University.

The series (1) is said to be φ-*absolute* A-*summable with the power* $p \geq 1$ or briefly, φ–$|A|_p$-*summable*, if (compare [7, 20])

$$\sum \varphi_n^{p-1} |v_n|^p < \infty. \tag{3}$$

Let B be a triangular method of summability, where by (β_{nk}) we denote the matrix of series-to-sequence transformation and by $(\bar{\beta}_{nk})$ the matrix of the series-to-series transformation. Let $\varepsilon = (\varepsilon_n)$ be a sequence of complex numbers, and $r, s \geq 1$. The numbers ε_n are called *summability factors of type* $(\varphi-|A|_r, \varphi-|B|_s)$, if for any $\varphi-|A|_r$-summable series (1) the series

$$\sum \varepsilon_n u_n \tag{4}$$

is $\varphi-|B|_s$-summable, that is

$$w_n := \sum_{k=0}^{n} \bar{\beta}_{nk} \varepsilon_k u_k \tag{5}$$

yields

$$\sum \varphi_n^{s-1} |w_n|^s < \infty$$

whenever (u_n) satisfies (3) (with $p = r$ in this case). We will use in the following the short notation: $\varepsilon \in (\varphi-|A|_r, \varphi-|B|_s)$, where we omit the index if it is 1. If $B = E$ is the convergence method, then summability factors are called *convergence factors* of the corresponding type. Problems of convergence factors were discussed for a general class of matrices in [5]. In this note we discuss *sufficient and necessary conditions* so that $\varepsilon \in (\varphi-|A|_r, \varphi-|B|_s)$ for the (r, s) combinations $(p, 1); (1, p); (p, p)$ for a real number $p > 1$ and for the class of so-called A^α-matrices, which are defined as follows [1, 2]: Let α be a non-negative integer. An A^α-method is called a normal method, whose series-to-series matrix $(\bar{\alpha}_{nk})$ has an inverse matrix $(\bar{\eta}_{nk})$ with exactly $\alpha + 1$ non-zero diagonals, namely $\bar{\eta}_{nk} = 0$ for $n < k$ and $n > k + \alpha$. The inverse matrix (η_{nk}) of the correspondig series-to-sequence matrix (α_{nk}) then satisfies $\eta_{nk} = \bar{\eta}_{nk} - \bar{\eta}_{n,k+1}$ and has $\alpha + 2$ non-zero diagonals.

Examples of A^α-methods:
(a) The Cesàro-methods C_γ of order $\gamma = 0, 1, 2 \ldots$ are A^γ-methods (see [3], p.86).
(b) The weighted means method M_a generated by a sequence $a = (a_n)$ with $a_n \neq 0$ is an A^1-method and the iteration product of two such methods is an A^2-method (see [2], p.177).
(c) The Nörlund-methods N_a with generating function

$$a(z) = \sum a_n z^n$$

which has a polynomial reciprocal function

$$c(z) := \frac{1}{a(z)} = \sum_{k=0}^{\alpha} c_k z^k \tag{6}$$

are A^α-methods, since $\eta_{nk} = A_k(c_{n-k} - c_{n-k-1})$, where $A_n := a_0 + \ldots a_n$ (see [3], p.103-104, and [16], p.17).
(d) Generalized Nörlund methods (N, a, b) are likewise A^α-methods if (6) is true (see [9, 11]).

In this paper we consider all the above mentioned types of summability factors, where A is an A^α-method and B satisfies some regularity conditions. Before stating our results we need a few notations. For the method A denote

$$\bar{D}_n := \sup_k |\alpha_{n+k,n+k} \bar{\eta}_{n+k,k}|, \quad G_k := \sum_{\nu=k}^{k+\alpha} \bar{\eta}_{\nu k} \varepsilon_\nu.$$

For the method B denote (cf. [3], p.52)

$$b_{nk} := \Delta\beta_{nk} := \beta_{nk} - \beta_{n,k+1}, \quad \bar{b}_{nk} := \Delta\bar{\beta}_{nk},$$
$$b(k,p) := \sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar{b}_{nk}\right|^p, \qquad \beta(k,p) := \sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar{\beta}_{nk}\right|^p.$$

As usual the number q is defined by the equation $1/p + 1/q = 1$. We also use Landau's $o(.)$ and $O(.)$ symbols, where the relevant index is supposed to tend to infinity. Let $p > 1$. We always assume *the existence of finite* $\bar{D}_n$ at $n = 1, 2, \ldots \alpha$ for the A^α-method and that B is a *normal* method (see [16], p.14).

THEOREM 1. *Assume that*

$$\varphi_{n+1} = O(\varphi_n) \tag{7}$$

and let the method B satisfy the conditions

$$\lim_{n\to\infty} \beta_{nk} = 1, \tag{8}$$
$$\beta_{kk} = O(\beta_{k+1,k+1}), \tag{9}$$
$$\beta(k,1) = O(1), \tag{10}$$
$$b(k,1) = O(\beta_{kk}). \tag{11}$$

Then $\varepsilon \in (\varphi{-}|A^\alpha|_p, |B|)$ if and only if the following conditions are satisfied:

$$\sum \varphi_n^{-1} |G_n|^q < \infty, \tag{12}$$
$$\sum \varphi_n^{-1} \left|\frac{\beta_{nn}\varepsilon_{nn}}{\alpha_{nn}}\right|^q < \infty. \tag{13}$$

Remark 1. The condition (8) is fulfilled if B is a regular matrix. By the Knopp-Lorentz theorem (see [3], p.34; [16], p.13) B preserves absolute convergence if and only if (10) is satisfied.

Examples of matrices B which satisfy (8)-(11) are
(a) Cesàro-methods C_β with $0 \le \beta \le 1$ (see [3], pp.85 and 188).
(b) M_a methods which preserve absolute convergence and are absolute translative from the right (see [3], pp.114 and 123; [2], p.176).
(c) N_a methods with decreasing, non-negative generating sequences $a = (a_n)$ (see [15], Corollary 1; [2], p.175).
(d) Generalized Nörlund methods (N, a, b) (see [9, 11]), where the generating sequence a is also assumed to be decreasing.

THEOREM 2. *Assume that φ satiesfies* (7) *and*

$$\varphi_n = O(\varphi_{n+1}) \tag{14}$$

and let the method B satisfy the conditions (9),

$$\beta_{k+1,k+1} = O(\beta_{kk}), \tag{15}$$
$$b(k,p) = O(\varphi_k^{p-1} |\beta_{kk}|^p). \tag{16}$$

Then $\varepsilon \in (|A^\alpha|, \varphi\text{–}|B|_p)$ *if and only if the following conditions*

$$\beta(k,p)\,|G_k|^p = O(1), \tag{17}$$
$$\beta_{kk}\varepsilon_k = O(\varphi_k^{-1/q}\alpha_{kk}) \tag{18}$$

are satisfied.

Examples of matrices B which fulfill (9), (15), (16):
(a) For C_β and $\varphi_n = n+1$ the conditions are satisfied when $-1 < \beta < 2 - 1/p$ (see [14], Lemma 5).
(b) M_a-means with regulary varying weights sequences $a = (a_n)$ and φ a sufficiently smooth sequence (compare §6).
(c) (N,a,b) methods with generating sequences $a_n = \binom{n+\alpha}{n}, \alpha \geq 0$, and $b_n n = O(B_n)$ (compare [11]).

For $\varepsilon \in (\varphi\text{–}|A^\alpha|_p, \varphi\text{–}|B|_p)$ we give *sufficient conditions* in our third theorem.

THEOREM 3. *Assume that* φ *satisfies* (7),(14) *and that the method* B *satisfies the conditions* (9), (15)*,*

$$\sum_{k=0}^{n} \varphi_k^{-1/q} \left|\frac{\bar{b}_{nk}}{\beta_{kk}}\right| = O(\varphi_n^{-1/q}), \tag{19}$$
$$\sum_{n=k}^{\infty} \varphi_n^{1/q} \left|\bar{b}_{nk}\right| = O(\beta_{kk}\varphi_k^{1/q}). \tag{20}$$

Then $\varepsilon \in (\varphi\text{–}|A^\alpha|_p, \varphi\text{–}|B|_p)$ *if*

$$\sum_{k=0}^{n} \varphi_k^{-1/q} \left|\bar{\beta}_{nk}G_k\right| = O(\varphi_n^{-1/q}), \tag{21}$$
$$\sum_{n=k}^{\infty} \varphi_n^{1/q} \left|\bar{\beta}_{nk}G_k\right| = O(\varphi_k^{1/q}), \tag{22}$$
$$\beta_{kk}\varepsilon_k = O(\alpha_{kk}) \tag{23}$$

are satisfied.

In the following three paragraphs we prove Theorems 1-3 by using sufficient and necessary conditions for matrix transforms from l^s into l^r as given in [8, 10, 12, 17]. Our main tool is the special structure of the A^α matrices. To show how useful this matrices are we state and prove a further Theorem in §5, giving *sufficient conditions* for $\varepsilon \in (\varphi\text{–}|A^\alpha|_p, \varphi\text{–}|B|_p)$. The type of these conditions has recently found wide use, see for example [7, 20, 21, 22] and the references therein. It turns out that by using our technique many of these results can also be obtained. As examples we provide summability factors for weighted means methods and the Silverman-Szazs methods.

2 Proof of Theorem 1

Set $x_n := \varphi_n^{1/q} v_n$ and w_n as in (5). Then using the inverse transformation of A we have

$$u_k = \sum_{\nu=0}^{k} \bar{\eta}_{k\nu} v_\nu = \sum_{\nu=0}^{k} \bar{\eta}_{k\nu} \varphi_\nu^{-1/q} x_\nu .$$

Therefore

$$w_n = \sum_{\nu=0}^{n} \bar{\beta}_{n\nu} \varepsilon_\nu \sum_{k=0}^{\nu} \bar{\eta}_{\nu k} \varphi_k^{-1/q} x_k = \sum_{k=0}^{n} x_k \varphi_k^{-1/q} \sum_{\nu=k}^{n} \bar{\beta}_{n\nu} \varepsilon_\nu \bar{\eta}_{\nu k},$$

and consequently

$$w_n = \sum_{k=0}^{n} \varphi_k^{-1/q} \bar{\gamma}_{nk} x_k,$$

where

$$\bar{\gamma}_{nk} := \sum_{\nu=k}^{n} \bar{\beta}_{n\nu} \varepsilon_\nu \bar{\eta}_{\nu k}.$$

Now we can use the necessary and sufficient conditions for matrices transforming l^p into l^1 (see [17], Theorem C). These conditions imply that $\varepsilon \in (\varphi\text{-}|A^\alpha|_p, |B|)$ if and only if

$$\sum_{k=0}^{\infty} \varphi_k^{-1} \left| \sum_{n\in K} \bar{\gamma}_{nk} \right|^q = O(1) \tag{24}$$

for each $K \in \mathcal{K}$, where $\mathcal{K}$ denotes the set of finite sets of distinct non-negative integers. Now (24) implies the necessity of

$$\sum_{k=0}^{l} \varphi_k^{-1} \left| \sum_{n=0}^{j} \bar{\gamma}_{nk} \right|^q = O(1) \quad \forall l, j = 0, 1, \ldots . \tag{25}$$

Since $\bar{\gamma}_{nk} = 0$ for $k > n$, we have for sufficiently large $j \geq k + \alpha + 1$ (see [3], p.51)

$$\begin{aligned} \sum_{n=0}^{j} \bar{\gamma}_{nk} &= \sum_{n=k}^{j} \sum_{\nu=k}^{n} \bar{\beta}_{n\nu} \varepsilon_\nu \bar{\eta}_{\nu k} = \sum_{\nu=k}^{j} \varepsilon_\nu \bar{\eta}_{\nu k} \sum_{n=\nu}^{j} \bar{\beta}_{n\nu} \\ &= \sum_{\nu=k}^{j} \varepsilon_\nu \bar{\eta}_{\nu k} \beta_{j\nu} = \sum_{\nu=k}^{k+\alpha} \varepsilon_\nu \bar{\eta}_{\nu k} \beta_{j\nu} \to G_k \quad (j \to \infty) \end{aligned}$$

by (8). Hence (25) implies (12). Furthermore we obtain (13) directly from (24) via Mehdi's Lemma (see [13], p.136, Lemma 12), which implies

$$\sum \varphi_n^{-1} |\bar{\gamma}_{nn}|^q < \infty$$

and this is just (13) as

$$\bar{\gamma}_{nn} = \frac{\bar{\beta}_{nn} \varepsilon_n}{\alpha_{nn}}.$$

To prove the sufficiency of (12) and (13) we show that these conditions yield

$$\sum_{k=0}^{\infty} \varphi_k^{-1} \sum_{n=k}^{\infty} |\bar{\gamma}_{nk}|^q < \infty, \tag{26}$$

which implies (24) because $(\bar{\gamma}_{nk})$ is a triangular matrix. By partial summation we obtain

$$\begin{aligned}\bar{\gamma}_{nk} &= \left(\sum_{\nu=k}^{n} \varepsilon_\nu \bar{\eta}_{\nu k}\right) \bar{\beta}_{n,n+1} + \sum_{\nu=k}^{n} \bar{b}_{n\nu} \sum_{\mu=k}^{\nu} \varepsilon_\mu \bar{\eta}_{\mu k} \\ &= \sum_{\nu=k}^{n} \bar{b}_{n\nu} \left(\sum_{\mu=k}^{k+\alpha} - \sum_{\mu=\nu+1}^{k+\alpha}\right) \varepsilon_\mu \bar{\eta}_{\mu k} = \sum_{\nu=k}^{n} \bar{b}_{n\nu} G_k - \sum_{\nu=k}^{n} \bar{b}_{n\nu} \sum_{\mu=\nu+1}^{k+\alpha} \varepsilon_\mu \bar{\eta}_{\mu k}.\end{aligned}$$

Hence we have

$$\bar{\gamma}_{nk} = \bar{\beta}_{nk} G_k - \bar{\lambda}_{nk}, \tag{27}$$

where

$$\bar{\lambda}_{nk} = \sum_{\nu=k}^{N} \bar{b}_{n\nu} \left(\sum_{\mu=\nu+1}^{k+\alpha} \varepsilon_\mu \bar{\eta}_{\mu k}\right), \quad N := \min\{n, k+\alpha-1\}. \tag{28}$$

Since B satisfies (10) the condition (12) yields

$$\sum_{k=0}^{\infty} \varphi_k^{-1} \left(\sum_{n=k}^{\infty} \left|\bar{\beta}_{nk} G_k\right|\right)^q = \sum_{k=0}^{\infty} \varphi_k^{-1} |G_k|^q \beta^q(k,1) = O(1) \sum_{k=0}^{\infty} \varphi_k^{-1} |G_k|^q < \infty.$$

In view of the latter relation, (27) and Minkowski's inequality, it remains to prove that

$$\sum_{k=0}^{\infty} \varphi_k^{-1} \left(\sum_{n=k}^{\infty} \left|\bar{\lambda}_{nk}\right|\right)^q < \infty.$$

Since B is triangular and satisfies (11) we have

$$\sum_{\nu=k}^{k+\alpha} \sum_{n=\nu}^{\infty} \left|\bar{b}_{n\nu}\right| = \sum_{\nu=k}^{k+\alpha} O(\beta_{\nu\nu})$$

and therefore by (9)

$$\sum_{n=k}^{\infty} \left|\bar{\lambda}_{nk}\right| = O(1) \sum_{\mu=k}^{k+\alpha} |\varepsilon_\mu \bar{\eta}_{\mu k} \beta_{\mu\mu}| = O(1) \sum_{\mu=0}^{\alpha} \bar{D}_\mu \left|\frac{\varepsilon_{k+\mu} \beta_{k+\mu,k+\mu}}{\alpha_{k+\mu,k+\mu}}\right|. \tag{29}$$

Hence by (7) and condition (13)

$$\sum_{k=0}^{\infty} \varphi_k^{-1} \left(\sum_{n=k}^{\infty} \left|\bar{\lambda}_{nk}\right|\right)^q = O(1) \sum_{\mu=0}^{\alpha} \sum_{k=\mu}^{\infty} \varphi_k^{-1} \left|\frac{\varepsilon_k \beta_{kk}}{\alpha_{kk}}\right|^q < \infty. \quad \blacksquare$$

Remark 2. For $\varphi_n = n+1$ this theorem was proved in [4]. For discussion of special methods and applications we also refer to this paper.

3 Proof of Theorem 2

We set $y_n = \varphi_n^{1/q} w_n$ then it follows, recall (5)

$$y_n = \varphi_n^{1/q} \sum_{k=0}^{n} \bar{\gamma}_{nk} v_k. \tag{30}$$

By applying the theorem of Cohen-Dunford (see [8], p.696, Theorem 10, or [10], p.349, Corollary 1.2, or [12], p.167, Theorem 5) to the transformation (30) we obtain that $\varepsilon \in (|A^\alpha|, \varphi{-}|B|_p)$ if and only if

$$\sup_k \sum_{n=k}^{\infty} \varphi_n^{p-1} |\bar{\gamma}_{nk}|^p < \infty. \tag{31}$$

By setting $n = k$ in (31) we deduce

$$\sup_k \varphi_k^{p-1} |\bar{\gamma}_{kk}|^p = \sup_k \varphi_k^{p-1} \left|\bar{\beta}_{kk}\varepsilon_k\bar{\eta}_{kk}\right|^p < \infty.$$

Hence we immediatly obtain (18). We postpone to show the necessity of (17) and turn to the sufficiency part. Recall the decomposition formula (27). We start by estimating $\left|\bar{\lambda}_{nk}\right|^p$ with the help of Hölders formula and (28) getting:

$$\left|\bar{\lambda}_{nk}\right|^p \le \left(\sum_{\nu=k}^{N} \left|\bar{b}_{n\nu}\right|^p\right) \lambda^{p/q}(k), \tag{32}$$

where

$$\lambda(k) := \sum_{\nu=k}^{N} \left|\sum_{\mu=\nu+1}^{k+\alpha} \bar{\eta}_{\mu k}\varepsilon_\mu\right|^q.$$

From (18), (14), (9) we get

$$\lambda(k) = O(1) \sum_{\mu=0}^{\alpha} \bar{D}_\mu \left|\varepsilon_{\mu+k}/\alpha_{\mu+k,\mu+k}\right|^q = O(\varphi_k^{-1} |\beta_{kk}|^{-q}).$$

Since B is normal and satisfies (15) and (16) we obtain by (7)

$$\sum_{n=k}^{\infty} \varphi_n^{p-1} \sum_{\nu=k}^{N} \left|\bar{b}_{n\nu}\right|^p \le \sum_{\nu=k}^{k+\alpha-1} b(\nu, p) = O(\varphi_k^{p-1} |\beta_{kk}|^p).$$

By combining these estimates with (32) we get

$$\sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar{\lambda}_{nk}\right|^p = O(\varphi_k^{-p/q} |\beta_{kk}|^{-p} \varphi_k^{p-1} |\beta_{kk}|^p) = O(1). \tag{33}$$

From (17) and (33) we deduce (31) by (27), since $|a+b|^p \le 2^p(|a|^p + |b|^p)$ for any real numbers a and b. The necessity of (17) now follows since the decomposition formula (27) implies

$$\bar{\beta}_{nk}G_k = \bar{\lambda}_{nk} + \bar{\gamma}_{nk}$$

and by Minkowski's inequality we obtain (17) vice versa from (31) and (33). ■

Remark 3. For $\varphi_n = n+1$ this Theorem was proved in [4].

4 Proof of Theorem 3

As in §2 resp. §3 we set $x_n = \varphi_n^{1/q} v_n$ resp. $y_n = \varphi_n^{1/q} w_n$. It follows from (30) that

$$y_n = \sum_{k=0}^{n} \left(\frac{\varphi_n}{\varphi_k}\right)^{1/q} \bar{\gamma}_{nk} x_k. \tag{34}$$

To prove that the transformation (34) preserves l^p-convergence, we have to show (see [12], p.174, Theorem 9) that

$$\sum_{k=0}^{n} \varphi_k^{-1/q} |\bar{\gamma}_{nk}| = O(\varphi_n^{-1/q}), \tag{35}$$

$$\sum_{n=k}^{\infty} \varphi_n^{1/q} |\bar{\gamma}_{nk}| = O(\varphi_k^{1/q}). \tag{36}$$

We start by proving (35). In view of the decomposition formula (27) and (21) it is only left to show that

$$\sum_{k=0}^{n} \varphi_k^{-1/q} \left|\bar{\lambda}_{nk}\right| = O(\varphi_n^{-1/q}). \tag{37}$$

By using (28) we get

$$\left|\bar{\lambda}_{nk}\right| \le \sum_{\mu=k}^{k+\alpha-1} \left|\bar{b}_{n\mu}\right| \sum_{\nu=\mu+1}^{k+\alpha} |\varepsilon_\nu \bar{\eta}_{\nu k}| = \sum_{\nu=k+1}^{k+\alpha} |\varepsilon_\nu \bar{\eta}_{\nu k}| \sum_{\mu=k}^{\nu-1} \left|\bar{b}_{n\mu}\right|$$

and by (23), (9), (15), (7) we obtain, since B is normal,

$$\begin{aligned}
\sum_{k=0}^{n} \varphi_k^{-1/q} \left|\bar{\lambda}_{nk}\right| &= O(1) \sum_{k=0}^{n} \varphi_k^{-1/q} \sum_{\nu=1}^{\alpha} \bar{D}_\nu \sum_{\mu=k}^{\nu+k-1} \left|\frac{\bar{b}_{n\mu}}{\beta_{k+\nu,k+\nu}}\right| \\
&= O(1) \sum_{k=0}^{n} \varphi_k^{-1/q} \left|\frac{\bar{b}_{nk}}{\beta_{kk}}\right|.
\end{aligned}$$

Now (37) follows from (19). To prove (36) in the view of (22) and (27) it only remains to show that

$$\sum_{n=k}^{\infty} \varphi_n^{1/q} \left|\bar{\lambda}_{nk}\right| = O(\varphi_k^{1/q}). \tag{38}$$

Now by (20), (9), (14), (23), (7) we get (38), in the same way as we obtained (29), namely

$$\begin{aligned}
\sum_{n=k}^{\infty} \varphi_n^{1/q} \left|\bar{\lambda}_{nk}\right| &= O(1) \sum_{\mu=k+1}^{k+\alpha} |\bar{\eta}_{\mu k} \varepsilon_\mu| \sum_{\nu=k}^{\mu-1} \sum_{n=\nu}^{\infty} \varphi_n^{1/q} \left|\bar{b}_{n\nu}\right| \\
&= O(1) \sum_{\mu=k+1}^{k+\alpha} |\bar{\eta}_{\mu k} \varepsilon_\mu b_{\mu\mu}| \, \varphi_\mu^{1/q} \\
&= O(1) \sum_{\mu=k+1}^{k+\alpha} |\bar{\eta}_{\mu k} \alpha_{\mu\mu}| \, \varphi_\mu^{1/q} = O(\varphi_k^{1/q}). \quad \blacksquare
\end{aligned}$$

Remark 4. We are not yet able to provide necessary conditions. The first author has observed that the proof of the necessity-part of Theorem 4 in [4] in incorrect.

5 Further sufficient conditions

THEOREM 4. *Assume that φ satisfies (7) and that the method B satisfies the conditions (15),*

$$\sum_{k=0}^{n} \left|\bar{\beta}_{nk}\right| = O(1), \tag{39}$$

$$\sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar{\beta}_{nk}\right| = O(\varphi_k^{p-1}), \tag{40}$$

$$\sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar{b}_{nk}\right| = O(\beta_{kk}\varphi_k^{p-1}) \tag{41}$$

Then sufficient conditions for $\varepsilon \in (\varphi\!-\!|A^\alpha|_p\,, \varphi\!-\!|B|_p)$ are

$$\sum_{n=0}^{\infty} \varphi_n^{p-1} \left|G_n\right|^p \left|v_n\right|^p < \infty, \tag{42}$$

$$\sum_{n=0}^{\infty} \varphi_n^{p-1} \left|\frac{\varepsilon_n \beta_{nn}}{\alpha_{nn}}\right| \left|v_n\right|^p < \infty, \tag{43}$$

$$\sum_{k=0}^{n} \left|\bar{\lambda}_{nk}\right| = O(1), \tag{44}$$

$$\frac{\varepsilon_{n+1}}{\alpha_{n+1,n+1}} = O\left(\frac{\varepsilon_n}{\alpha_{nn}}\right). \tag{45}$$

where the numbers $\bar{\lambda}_{nk}$ are defined by (28) and v_n is defined by (2) and satisfies (3).

Proof. As in the proof of Theorem 1 we have

$$w_n = \sum_{k=0}^{n} \bar{\gamma}_{nk} v_k.$$

We must prove that (3) implies

$$\sum \varphi_n^{p-1} \left|w_n\right|^p < \infty.$$

By using the decomposition formula (27) and $|u+v|^p \leq 2^p(|u|^p + |v|^p)$ it suffices to show that

$$I_1 := \sum_{n=0}^{\infty} \varphi_n^{p-1} \left|\sum_{k=0}^{n} \bar{\beta}_{nk} G_k v_k\right|^p < \infty$$

and

$$I_2 := \sum_{n=0}^{\infty} \varphi_n^{p-1} \left|\sum_{k=0}^{n} \bar{\lambda}_{nk} v_k\right|^p < \infty.$$

By Hölder's inequality, (39), (40), (42) we obtain

$$\begin{aligned} I_1 &\leq \sum_{n=0}^{\infty} \varphi_n^{p-1} \left(\sum_{k=0}^{n} \left|\bar\beta_{nk}\right| |G_k|^p \, |v_k|^p \right) \left(\sum_{k=0}^{n} \left|\bar\beta_{nk}\right| \right)^{p/q} \\ &= O(1) \sum_{k=0}^{\infty} |G_k|^p \, |v_k|^p \sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar\beta_{nk}\right| = O(1) \sum_{k=0}^{\infty} \varphi_k^{p-1} \, |G_k|^p \, |v_k|^p < \infty. \end{aligned}$$

Again by Hölder's inequality and now by (44) we get

$$\begin{aligned} I_2 &\leq \sum_{n=0}^{\infty} \varphi_n^{p-1} \left(\sum_{k=0}^{n} \left|\bar\lambda_{nk}\right| |v_k|^p \right) \left(\sum_{k=0}^{n} \left|\bar\lambda_{nk}\right| \right)^{p/q} \\ &= O(1) \sum_{k=0}^{\infty} |v_k|^p \sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar\lambda_{nk}\right|. \end{aligned}$$

By (28) and (41) we obtain

$$\begin{aligned} \sum_{n=k}^{\infty} \varphi_n^{p-1} \left|\bar\lambda_{nk}\right| &\leq \sum_{n=k}^{k+\alpha-1} \varphi_n^{p-1} \sum_{\nu=k}^{n} \left|\bar b_{n\nu}\right| \left(\sum_{\mu=\nu+1}^{k+\alpha} |\bar\eta_{\mu k} \varepsilon_\mu| \right) \\ &\quad + \sum_{n=k+\alpha}^{\infty} \varphi_n^{p-1} \sum_{\nu=k}^{k+\alpha-1} \left|\bar b_{n\nu}\right| \left(\sum_{\mu=\nu+1}^{k+\alpha} |\bar\eta_{\mu k} \varepsilon_\mu| \right) \\ &= O(1) \sum_{\nu=k}^{k+\alpha} \varphi_\nu^{p-1} \, |\beta_{\nu\nu}| \left(\sum_{\mu=\nu+1}^{k+\alpha} |\bar\eta_{\mu k} \varepsilon_\mu| \right). \end{aligned}$$

From this using (7), (15), (45), (43) we obtain

$$\begin{aligned} I_2 &= O(1) \sum_{k=0}^{\infty} |v_k|^p \sum_{\nu=k}^{k+\alpha} \varphi_\nu^{p-1} \, |\beta_{\nu\nu}| \sum_{\mu=\nu+1}^{k+\alpha} |\bar\eta_{\mu k} \varepsilon_\mu| \\ &= O(1) \sum_{k=0}^{\infty} |v_k|^p \, \varphi_k^{p-1} \, |\beta_{kk}| \sum_{\mu=k+1}^{k+\alpha} |\bar\eta_{\mu k} \varepsilon_\mu| \\ &= O(1) \sum_{k=0}^{\infty} |v_k|^p \, \varphi_k^{p-1} \left| \frac{\beta_{kk} \varepsilon_k}{\alpha_{kk}} \right| \sum_{\mu=1}^{\alpha} \bar D_\mu < \infty. \quad \blacksquare \end{aligned}$$

6 Applications to special methods

We now use Theorem 4 to obtain summability factors for special methods.

Let $A = M_a$ and $B = M_b$ with $a_n \sim n^\alpha L_1(n)$ and $b_n \sim n^\beta L_2(n)$, where $\alpha, \beta > -1$ and the functions L_1 and L_2 are slowly varying (see [6], p.6 for the definition and relevant properties). Let $\varphi_n \sim n^\gamma L_3(n), \gamma > 0$, and L_3 be also a slowly varying function. As usual

$$A_n := \sum_{k=0}^{n} a_k, \quad B_n := \sum_{k=0}^{n} b_k.$$

Then we have the following

COROLLARY 1. *For the weighted means methods M_a and M_b the conditions*

$$\frac{A_n b_n}{B_n a_n}\varepsilon_n = O(1), \tag{46}$$

$$\varepsilon_n = O(1), \tag{47}$$

$$\frac{A_n}{a_n}\Delta\varepsilon_n = O(1) \tag{48}$$

are sufficient for $\varepsilon \in (\varphi-|M_a|_p, \varphi-|M_b|_p)$ provided that the sequence φ is, as above, with $\gamma(p-1) < \beta+1$.

Proof. The methods M_a are A^1 methods (see [3], p.116). Furthermore (see [3], p.113)

$$\bar\beta_{nk} = \begin{cases} \dfrac{b_n B_{k-1}}{B_n B_{n-1}}, & 1 \le k \le n, \\ \\ 0, & \text{otherwise,} \end{cases}$$

and (see [3], p.184, and [4], p.201)

$$G_n = \frac{A_n}{a_n}\varepsilon_n - \frac{A_{n-1}}{a_n}\varepsilon_{n+1}.$$

Since (B_{k-1}) is non-decreasing condition (39) follows. Furthermore $B_n \sim (\beta+1)^{-1} n^{\beta+1} L_2(n)$ (see [6], Proposition 1.5.8, p.26) and hence (15) is satisfied. The fulfilment of the conditions (40) and (41) is seen by applying Proposition 1.5.10 in [6], p.27 (or cf. [3], pp.80-81 in particular cases), which implies for $\gamma(p-1) < \beta+1$ the following formula

$$\sum_{n=k}^{\infty} n^{\gamma(p-1)-\beta-2}\frac{L_3^{p-1}(n)}{L_2(n-1)} = O(1)k^{\gamma(p-1)-\beta-1}\frac{L_3^{p-1}(k)}{L_2(k-1)}.$$

Hence for $\gamma(p-1) < \beta+1$ our Theorem 4 is applicable for M_b. From (48) and (47) we see that

$$|G_n| = \left|\frac{A_n}{a_n}\Delta\varepsilon_n\right| + |\varepsilon_{n+1}| = O(1)$$

and by (3) we obtain (42). Furthermore (43) is directly implied by (46) and (3). Using (46),

(47) and again the asymptotics of regularly varying functions it follows that

$$\begin{aligned}
\sum_{k=0}^{n}\left|\bar\lambda_{nk}\right| &\le \sum_{k=0}^{n}\left|\sum_{\nu=k}^{n}\bar b_{n\nu}\left(\sum_{\mu=\nu+1}^{k+\alpha}\varepsilon_\mu\bar\eta_{\mu k}\right)\right| = \sum_{k=0}^{n}\left|\sum_{\nu=k}^{n}\frac{-b_n b_\nu}{B_nB_{n-1}}\sum_{\mu=\nu+1}^{k+1}\varepsilon_\mu\bar\eta_{\mu k}\right| \\
&= \frac{b_n}{B_nB_{n-1}}\sum_{k=0}^{n}|b_k\varepsilon_{k+1}\bar\eta_{k+1,k}| = \frac{b_n}{B_nB_{n-1}}\sum_{k=0}^{n}O(B_k) \\
&= O(1)\frac{nb_n}{B_n} = O(1),
\end{aligned}$$

since by (47), (48), (46) we obtain

$$\begin{aligned}
\varepsilon_{k+1}\bar\eta_{k+1,k} &= -\frac{\varepsilon_{k+1}A_{k-1}}{a_k} = G_k - \frac{\varepsilon_k A_k}{a_k} \\
&= O(1) + O(B_k/b_k) = O(B_k/b_k).
\end{aligned}$$

By (47) and the special choice of weights (45) is obtained in the same way. By Theorem 4 the collorary is proved.■

Remark 5.

(i) The above Corollary 1 contains certain cases of summability methods discussed in [21], especially Corollary 4, p.317.

(ii) One can also use Theorem 4 resp. Corollary 1 to obtain Theorem 1 in [7], p.146. There summability factors of type $\varepsilon \in (\varphi-|M_a|_p, \psi-|M_b|_p)$ are considered. The condition (6) on p.146 in [7], however, allows to switch from the sequence φ to ψ and vice versa and allows Theorem 4 to be applied.

For a second application we consider the method $B = Z_\beta$ of Silverman-Szasz (see [18], p.347, [3], p.100). We have $b_{nk} = a_{n-k}/A_n$, where $a_n = 1$ for $n < \beta$ and $a_n = 0$ for $n \ge \beta$. Hence $A_n = n+1$ for $n < \beta$ and $A_n = \beta$ for $n \ge \beta$. Therefore

$$1/\beta \le \beta_{kk} \le 1 \tag{49}$$

and condition (9) is satisfied. Since $b_{nk} = 0$ for $k \le n-\beta$, then $\bar b_{nk} = 0$ for $k \le n-\beta$ and $n \ge k+\beta+1$ (see [3], p.50). Therefore the conditions (19) and (20) follow from (7). By (49) we obtain that (23) implies

$$G_k = \sum_{\nu=0}^{\alpha}|\bar\eta_{\nu+k,k}|O(\alpha_{\nu+k,\nu+k}) = O(1)\sum_{\nu=0}^{\alpha}\bar D_\nu = O(1),$$

and consequently the conditions (21) and (22) follow from (23) and (7), because

$$\bar\beta_{nk} = A_{n-k}\bar\Delta(1/A_n) + (1/A_{n-1})\bar\Delta A_{n-k}$$

(see [3], p.51 and p.96) and therefore $\bar\beta_{nk} = 0$ for $k \le n-\beta$ and $n \ge k+1+\beta$. The necessity of (23) follows from a result of Jakimovski and Russell (see [10], Theorem 1 and formulas (5) and (17) with $x_k = \delta_{k\nu}$). Thus from Theorem 3 we obtain the following

COROLLARY 2. *Assume that* φ *satisfies* (7) *and* (14). *Then* $\varepsilon \in (\varphi\text{–}|A^\alpha|_p, \varphi\text{–}|Z_\beta|_p)$ *if and only if*

$$\varepsilon_k = O(\alpha_{kk})$$

is satisfied.

In particular for the convergence method $E = Z_1$ we have a related result (see [5], Theorem 2).

REFERENCES

1. Abel, M., Türnpu, H.: On ψ-convergence factors. Tartu Ülik. Toimetised 206, 106-121 (1967) (in Russian).
2. Baron, S.: Theorems on summability factors for A^α methods. Tartu Ülik. Toimetised 253, 165-178 (1970) (in Russian).
3. Baron, S.: Introduction to the Theory of Summability of Series. (Second Edition, corrected and supplemented), Tallinn, Valgus 1977 (in Russian).
4. Baron, S.: Absolute summability factors with a power for A^α-methods. Analysis 5, 197-208 (1985).
5. Baron, S., Kiesel, R.: Absolute φ-convergence factors with a power. J. of Analysis 2, 115-122 (1994).
6. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press 1987.
7. Bor, H., Thorpe, B.: On some absolute summability methods. Analysis 7, 145-152 (1987).
8. Cohen, L.W., Dunford, N.: Transformations on sequence spaces. Duke Math. J. 3, 689-701 (1937).
9. Das, G.: On some methods of summability. Quart. J. Math. 17, 244-256 (1966).
10. Jakimovski, A., Russel, D.: Matrix mappings between *BK*-spaces. Bull. London Math. Soc. 4, 345-353 (1972).
11. Kiesel, R.: On scales of summability methods, to appear in Math. Nachr.
12. Maddox, I.J.: Elements of Functional Analysis. Cambridge University Press 1970.
13. Mehdi, M.R.: Linear transformations between the Banach spaces L^p and l^p with applications to absolute summability. Dissertation, London 1959.
14. Medhi, M.R.: Summability factors for generalized absolute summability. Proc. London Math. Soc. 10, 180-200 (1960).

15. Mears, F.M.: Absolute regularity and the Nörlund mean. Ann. Math.(2) 38, 594-601 (1937).

16. Peyerimhoff, A.: Lectures on summability. Lecture Notes in Mathematics 107, Springer Verlag 1969.

17. Sargent, W.L.C.: Some sequences related to the l^p spaces. J. London Math. Soc. 35, 161-171 (1960).

18. Silverman, L.L.; Szasz, O.: On a class of Nörlund matrices. Ann. Math. 45, 347-357 (1944).

19. Sinha, R.: Convexity theorems for (N, p, q) summability. Kyungpook Math. J. 13, 37-40 (1973).

20. Sulaiman, W.T.: A study on relations between two summability methods. Proc. Amer. Math. Soc. 115, 303-312 (1992).

21. Sulaiman, W.T.: On some summability factors of infinite series. Proc. Amer. Math. Soc. 115, 313-317 (1992).

22. Sulaiman, W.T.: Relations on some summability methods. Proc. Amer. Math. Soc. 118, 1139-1145 (1993).

Simson Baron and Rüdiger Kiesel
Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel,
Department of Stochastics, University of Ulm, 89069 Ulm, Germany.