Absolute φ-convergence factors with a power

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We give necessary and sufficient conditions for a sequence (ε_n) of complex numbers to be convergence factors of type $(\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$, where A is a normal matrix of summability satisfying the Moore-Kangro condition and E is the convergence matrix.

1 Introduction and main result

Let $\varphi = (\varphi_n)$ be a sequence of positive numbers. The series

$$\sum u_n \tag{1}$$

of complex numbers u_n is said to be φ*-absolute convergent with the power* $p \geq 1$ or briefly, $\varphi\text{–}|E|_p$ *-convergent*, if

$$\sum \varphi_n^{p-1} |u_n|^p < \infty. \tag{2}$$

Unless otherwise indicated the free indices always run through all values $0, 1, 2, \ldots$ and summation is likewise over $0, 1, 2, \ldots$.

Let A be a triangular method of summability with the matrix (α_{nk}) of series-to-sequence transformation and with the matrix $(\bar{\alpha}_{nk})$ of series-to-series transformation. Let

$$v_n = \sum_{k=0}^{n} \bar{\alpha}_{nk} u_k. \tag{3}$$

[1]AMS-Subject classification: 40 F

The series (1) is called *φ-absolute A-summable with the power $p \geq 1$*, or briefly *φ–$|A|_p$-summable*, if (see [14], p.314)

$$\sum \varphi_n^{p-1} |v_n|^p < \infty. \tag{4}$$

In particular, if $A = E$, the identity matrix, (4) is the same as (2).

Let $\varepsilon = (\varepsilon_n)$ be a sequence of complex numbers. The numbers ε_n are called *convergence factors of type* $(\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$, if for any φ–$|A|_p$-summable series (1) the series

$$\sum \varepsilon_n u_n \tag{5}$$

is φ–$|E|_p$-convergent. The short notation is: $\varepsilon \in (\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$.

In this note we discover sufficient conditions such that $\varepsilon \in (\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$ for $p > 1$ for a general class of matrices A, namely normal matrices satisfying the so-called Moore-Kangro condition, (see [3], p.190),

$$\sum nD_n < \infty, \tag{6}$$

where

$$D_n := \sup_k |\alpha_{n+k,n+k}\eta_{n+k,k}|$$

and $(\eta_{nk}) := (\alpha_{nk})^{-1}$ denotes the inverse series-to-sequence transform (which exits since A is normal).

To find such conditions one can use the theory of matrix transformations from l^p to l^p. For such transformations *sufficient conditions* are known ([11], p.174, Theorem 9; [12], pp. 55-56), namely: a triangular matrix $G = (\bar{\gamma}_{nk})$ transforms any sequence $(x_n) \in l^p$ in a sequence $(y_n) \in l^p$ if

$$\sup_{n \geq 0} \sum_{k=0}^{n} |\bar{\gamma}_{nk}| < \infty \tag{7}$$

and

$$\sup_{k \geq 0} \sum_{n=k}^{\infty} |\bar{\gamma}_{nk}| < \infty. \tag{8}$$

On the other hand *necessary conditions* are also known ([9], Theorem 1 and formulas (5),(17) with $x_k = \delta_{k\nu}$; and [7], p.467), namely

$$\sup_{n \geq 0} \sum_{k=0}^{n} |\bar{\gamma}_{nk}|^q < \infty \tag{9}$$

and

$$\sup_{k \geq 0} \sum_{n=k}^{\infty} |\bar{\gamma}_{nk}|^p < \infty, \tag{10}$$

where q denotes always the conjugate exponent, i.e. $1/p + 1/q = 1$.

In the following we use the notations: $m := [n/2]$ and

$$\eta_n := \sum_{k=0}^{n} \eta_{nk}.$$

We now formulate our main theorem.

THEOREM 1. *Let A be a normal method of summability satisfying the condition (6) and let $\varphi = (\varphi_n)$ be nondecreasing and satisfy the conditions*

$$\varphi_n = O(\varphi_k), \qquad m < k \le n, \tag{11}$$

$$\frac{\varphi_n^{1/q}}{n} = O\left(\frac{\varphi_k^{1/q}}{k}\right), \quad 1 \le k \le n, \tag{12}$$

$$\sum_{k=0}^{m} \varphi_k^{-1/q} = O(n\varphi_n^{-1/q}). \tag{13}$$

Then the numbers ε_n are convergence factors of type $(\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$ for $p > 1$, if the following conditions are fullfilled:

$$\varepsilon_n = O(\alpha_{nn}), \tag{14}$$

$$n\varepsilon_n \eta_n = O(1), \tag{15}$$

$$\sum_{n=k}^{\infty} \varphi_n^{1/q} |\varepsilon_n \eta_n| = O(\varphi_k^{1/q}). \tag{16}$$

Remark 1.(a) For the special sequence $\varphi_n = n$ with $n \ge 1$ and $\varphi_0 = 1$ Theorem 1 is proved in [2].

(b) As choices for φ_n (satisfying (11)-(13)) sequences like $n^\gamma L(n)$ with $0 < \gamma < q$ and L a slowly varying function are always possible, i.e. L is a positive measurable function, defined on some neighbourhood $[X, \infty)$ of infinity, and satisfies

$$L(\lambda x)/L(x) \to 1 \quad (x \to \infty) \quad \forall \lambda > 0$$

(see [5], p.6 for further information). Especially $\varphi_n = (n+1)\log(n+1)$ can always be used, since $q > 1$.

Remark 2.(a) Riesz-Means M_a generated by a sequence $a = (a_n)$ of nonnegative numbers, with $A_n := a_0 + \ldots + a_n \to \infty$ as $n \to \infty$ satisfy (6),(15),(16), since $D_n = \eta_n = 0 \quad \forall n \ge 2$, (compare [3], p.58 and p.116). For such methods (14) reduces to

$$\varepsilon_n = O(a_n/A_n).$$

In case $a_n = n^\gamma L(n)$ with $\gamma > -1$ and L slowly varying, we have $A_n \sim (\gamma + 1)^{-1} n^{\gamma+1} L(n)$. Therefore the choice $\varphi_n = A_n/a_n$ is possible and we obtain the so-called $|M_a|_p$-methods, (see [6], p.40).

(b) Cesàro-methods C_α satisfy (6), (15), (16) for $\alpha \geq 0$ (cf. [3], p.58 and p.192) and (14) is in this case

$$\varepsilon_n = O(n^{-\alpha}).$$

(c) For Nörlund methods N_b generated by a sequence $b = (b_n)$ with $B_n := b_0 + b_1 + \ldots + b_n$ satisfying $B_k = O(B_n)$, $k \leq n$, we have $D_n = O(d_n)$, where

$$\sum d_n x^n = 1 / \sum B_n x^n$$

and the condition (6) reduces to $\sum n\,|d_n| < \infty$ (cf. [13], p.44).

(d) Generally one can use A^α-methods as introduced in [1], p.108, that is methods A for which $\eta_{nk} = 0$ for $k < n - \alpha - 1$. This methods always satisfy (6) and so only (14),(15),(16) have to be verified. A large class of such methods is given by the generalized Nörlund methods (N, a, b), (see [8, 10]), for which the series

$$\sum c_n x^n = 1 / \sum a_n x^n$$

reduces to a polynomial (see the special cases above).

In the next section we start by giving a proof of Theorem 1. In the third section we discuss conditions on the matrix A so that (14) becomes a necessary and sufficient condition for $\varepsilon \in (\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$.

2 Proof of Theorem 1

Denote $\alpha := 1/p$ and $\beta := 1/q$, furthermore $x_n := \varphi_n^\beta v_n$ and $y_n := \varphi_n^\beta \varepsilon_n u_n$. From (3) we obtain

$$y_n = \sum_{k=0}^{n} \bar{\gamma}_{nk} x_k, \tag{17}$$

where

$$\bar{\gamma}_{nk} = (\varphi_n/\varphi_k)^\beta \varepsilon_n \bar{\eta}_{nk}.$$

Since $|y_n|^p = \varphi_n^{p-1} |\varepsilon_n u_n|^p$ the series (5) is $\varphi\text{–}|E|_p$-convergent and therefore we have $\varepsilon \in (\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$ if and only if the matrix transformation (17) preserves the space l^p.

We begin by checking the condition (7). By formula (9.6) from [3] we obtain

$$\begin{aligned} \bar{\gamma}_{nk} &= (\varphi_n/\varphi_k)^\beta \varepsilon_n \left(\eta_n - \sum_{\nu=0}^{k-1} \eta_{n\nu} \right), \quad k \geq 1, \\ \bar{\gamma}_{n0} &= (\varphi_n/\varphi_0)^\beta \varepsilon_n \eta_n. \end{aligned} \tag{18}$$

By (13), (11), (15) we obtain from this formula

$$\sum_{k=0}^{n} |\bar{\gamma}_{nk}| \leq \varphi_n^\beta \, |\varepsilon_n \eta_n| \sum_{k=0}^{n} \varphi_k^{-\beta} + \varphi_n^\beta B_n = O(1) + \varphi_n^\beta B_n$$

where

$$B_n := |\varepsilon_n| \sum_{k=1}^{n} \varphi_k^{-\beta} \sum_{\nu=0}^{k-1} |\eta_{n\nu}| \, .$$

By (14) we obtain

$$\begin{aligned} B_n &= O(1) \sum_{k=0}^{n} \varphi_k^{-\beta} \sum_{\nu=0}^{k} |\alpha_{nn} \eta_{n\nu}| \\ &= O(1) \sum_{k=0}^{n} \varphi_k^{-\beta} \sum_{\nu=0}^{k} D_{n-\nu} \\ &= O(1) \sum_{k=0}^{n} \varphi_k^{-\beta} \sum_{\nu=n-k}^{n} D_\nu = O(1)(C_n + E_n), \end{aligned}$$

where, since $m = [n/2]$

$$C_n := \sum_{k=0}^{m} \varphi_k^{-\beta} \sum_{\nu=n-k}^{n} D_\nu, \quad E_n := \sum_{k=m}^{n} \varphi_k^{-\beta} \sum_{\nu=n-k}^{n} D_\nu.$$

By (6) and (13) we obtain for $n \geq 1$

$$\begin{aligned} \varphi_n^\beta C_n &= O(1/n) \varphi_n^\beta \sum_{k=1}^{m} \varphi_k^{-\beta} \sum_{\nu=n-k}^{n} \nu D_\nu \\ &= O(1/n) \varphi_n^\beta \sum_{k=1}^{m} \varphi_k^{-\beta} = O(1) \end{aligned}$$

and (11) and (6) give

$$\begin{aligned} \varphi_n^\beta E_n &= \varphi_n^\beta \sum_{k=m}^{n} \varphi_k^{-\beta} \sum_{\nu=n-k}^{n} D_\nu = O(1) \sum_{k=m}^{n} \sum_{\nu=n-k}^{n} D_\nu \\ &= O(1) \sum_{\nu=0}^{n} \nu D_\nu = O(1). \end{aligned}$$

Thus (7) follows from (6),(11),(13),(14),(15).

To establish condition (8) we have, in view of (18), to show that

$$\sum_{n=k}^{\infty} \varphi_n^{\beta} \left(|\varepsilon_n \eta_n| + \left| \varepsilon_n \sum_{\nu=0}^{k-1} \eta_{n\nu} \right| \right) = O(\varphi_k^{\beta}), \quad k \geq 1.$$

By (16) it only remains to show that

$$F_k = O(\varphi_k^{\beta}), \quad k \geq 1,$$

where

$$F_k := \sum_{n=k}^{\infty} \varphi_n^{\beta} \left| \varepsilon_n \sum_{\nu=0}^{k-1} \eta_{n\nu} \right|.$$

Denote

$$\psi_n := \varphi_n^{\beta} - \varphi_{n-1}^{\beta},$$

with $\varphi_{-1} := 0$. Since φ is nondecreasing, we have $\psi_n \geq 0$, and therefore

$$F_k \leq \sum_{n=k}^{\infty} \sum_{j=0}^{n} \psi_j \sum_{\nu=0}^{k} |\varepsilon_n \eta_{n\nu}| = O(1)(G_k + H_k),$$

where

$$G_k := \sum_{\nu=0}^{k} \sum_{j=0}^{k} \psi_j \sum_{n=k}^{\infty} |\varepsilon_n \eta_{n\nu}|, \quad H_k := \sum_{\nu=0}^{k} \sum_{j=k}^{\infty} \psi_j \sum_{n=j}^{\infty} |\varepsilon_n \eta_{n\nu}|.$$

From (14) and (6) we get

$$\begin{aligned} G_k &= \sum_{\nu=0}^{k} \sum_{j=0}^{k} \psi_j \sum_{n=k-\nu}^{\infty} |\varepsilon_{n+\nu} \eta_{n+\nu,\nu}| = O(1) \sum_{j=0}^{k} \psi_j \sum_{\nu=0}^{k} \sum_{n=k-\nu}^{\infty} D_n \\ &= O(1) \sum_{j=0}^{k} \psi_j \left(\sum_{n=0}^{k} D_n \sum_{\nu=k-n}^{k} 1 + \sum_{n=k+1}^{\infty} D_n \sum_{\nu=0}^{k} 1 \right) \\ &= O(1) \sum_{j=0}^{k} \psi_j \sum_{n=0}^{\infty} (n+1) D_n = O(1) \sum_{j=0}^{k} \psi_j = O(\varphi_k^{\beta}). \end{aligned}$$

Finally by (14)

$$H_k = \sum_{j=k}^{\infty} \psi_j \sum_{\nu=0}^{k} \sum_{n=j-\nu}^{\infty} |\varepsilon_{n+\nu} \eta_{n+\nu,\nu}| = O(1)(K_k + L_k),$$

where

$$\begin{aligned} K_k &:= \sum_{j=k}^{\infty} \psi_j \sum_{n=j-k}^{j} D_n \sum_{\nu=j-n}^{k} 1 = \sum_{j=k}^{\infty} \psi_j \sum_{n=j-k}^{j} D_n (k+n-j+1), \\ L_k &:= \sum_{j=k}^{\infty} \psi_j \sum_{n=j+1}^{\infty} D_n \sum_{\nu=0}^{k} 1 \leq \sum_{j=k}^{\infty} \psi_j \sum_{n=j}^{\infty} (k+1) D_n. \end{aligned}$$

From (12) and (6) follows

$$\begin{aligned}
K_k &= \sum_{n=0}^{k} D_n \sum_{j=k}^{n+k} (n+k-j+1)\psi_j + \sum_{n=k+1}^{\infty} D_n \sum_{j=n}^{n+k} (n+k-j+1)\psi_j \\
&\leq \sum_{n=0}^{k} (n+1) D_n \sum_{j=k}^{n+k} \psi_j + (k+1) \sum_{n=k}^{\infty} D_n \sum_{j=n}^{n+k} \psi_j \\
&= \sum_{n=0}^{k} (n+1) D_n (\varphi_{n+k}^{\beta} - \varphi_{k-1}^{\beta}) + (k+1) \sum_{n=k}^{\infty} D_n (\varphi_{n+k}^{\beta} - \varphi_{n-1}^{\beta}) \\
&= \sum_{n=0}^{k} (n+1)(n+k) D_n O\left(\frac{\varphi_k^{\beta}}{k}\right) + (k+1) \sum_{n=k}^{\infty} (n+k) D_n O\left(\frac{\varphi_k^{\beta}}{k}\right) \\
&= O\left(\varphi_k^{\beta}\right) \sum_{n=0}^{k} (n+1) D_n + O\left(\varphi_k^{\beta}\right) \sum_{n=0}^{k} 2n D_n = O(\varphi_k^{\beta}).
\end{aligned}$$

Also from (12) and (6) we obtain

$$\begin{aligned}
L_k &= (k+1) \sum_{n=k}^{\infty} D_n \sum_{j=k}^{n} \psi_j = (k+1) \sum_{n=k}^{\infty} D_n \left(\varphi_n^{\beta} - \varphi_{k-1}^{\beta}\right) \\
&= O(1)(k+1) \sum_{n=k}^{\infty} n D_n \frac{\varphi_k^{\beta}}{k} = O(\varphi_k^{\beta}).
\end{aligned}$$

Thus Theorem 1 is proved ∎

3 Necessarity of the conditions

Theorem 2. *Let φ be nondecreasing and satisfy the conditions* (11),(12),(13) *and the method of summability A satisfy* (6) *and $\alpha_{n0} = 1$. Then the numbers ε_n are convergence factors of type $(\varphi\text{–}|A|_p, \varphi\text{–}|E|_p)$ for $p > 1$ if and only if condition* (14) *is fullfilled.*

Proof. If $\alpha_{n0} = 1$, then $\eta_n = \delta_{n0}$ (see [3], p.58, Theorem 9.2). Hence conditions (15) and (16) are automatically true. Furthermore by ([3], p.57) we have

$$\bar{\gamma}_{nn} = \varepsilon_n \eta_{nn} = \varepsilon_n / \alpha_{nn}$$

and we can use (9) or (10) to obtain the necessarity of (14). By Theorem 1 we get the sufficiency of (14). ∎

Remark 3. With the exception of the general A^{α} methods all examples above obey the restriction $\alpha_{n0} = 1$.

Remark 4. For the case $\varphi_n = n,\ n \geq 1$ and $\varphi_0 = 1$ Theorem 2 is proved in [2].

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