Tauberian- and Convexity theorems for certain (N,p,q)–means

by

Rüdiger Kiesel and Ulrich Stadtmüller

1 Introduction

We consider throughout complex sequences (s_n) and discuss the relations of certain summability methods.
We say a sequence (s_n) of complex numbers is summable to s by the

(i) **Cesàro-method** of order $\alpha > -1$, briefly $s_n \to s(C_\alpha)$, if

$$\frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} s_k \to s \quad (n \to \infty);$$

(ii) **Euler-method** of order $0 < p \le 1$, briefly $s_n \to s(E_p)$, if

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s_k \to s \quad (n \to \infty);$$

(iii) **Abel-method**, briefly $s_n \to s(A)$, if

$$f(t) = (1-t) \sum_{n=0}^{\infty} s_n t^n \text{ exists for } \quad 0 < t < 1 \quad \text{and} \quad f(t) \to s \quad (t \to 1-);$$

(iv) **Borel-method**, briefly $s_n \to s(B)$, if

$$g(t) = e^{-t} \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n \text{ exists for } \quad t \in I\!R \text{ and } \quad g(t) \to s \quad (t \to \infty).$$

The Cesàro- and Abel-method resp. the Euler- and Borel-method are known to be closely related, see [9,17,19].
Especially the following Abelian inclusions are well kown, see e.g. [9; Theorems 43,55,118,128]

for $-1 < \alpha \le \beta$: $\quad s_n \to s(C_\alpha) \quad \Rightarrow \quad s_n \to s(C_\beta) \quad \Rightarrow \quad s_n \to s(A),$

for $0 < p \le q \le 1$: $\quad s_n \to s(E_q) \quad \Rightarrow \quad s_n \to s(E_p) \quad \Rightarrow \quad s_n \to s(B).$

The following converse or Tauberian theorem for the Cesàro–Abel-case goes back to Littlewood [14] ($\alpha, \beta \in I\!N$), and Anderson [1] ($\alpha, \beta \geq -1$).

THEOREM TC 1

(i) Let $-1 < \alpha < \beta$ *then*

$$s_n \to s(A) \ \ \textit{and} \quad s_n = \boldsymbol{O}\,(1)\,(C_\alpha) \quad \textit{imply} \ \ s_n \to s(C_\beta)\,.$$

(ii) For $-1 < \alpha < \delta \leq \beta$ *we have the so-called* **convexity-theorem**

$$s_n \to s(C_\beta) \ \ \textit{and} \quad s_n = \boldsymbol{O}\,(1)\,(C_\alpha) \quad \textit{imply} \ \ s_n \to s(C_\delta)\,.$$

Quite recently Boos and Tietz [4] proved that the situation is completely analogous for the Euler–Borel-case

THEOREM TC 2

(i) Let $0 < p < q \leq 1$ *then*

$$s_n \to s(B) \ \ \textit{and} \quad s_n = \boldsymbol{O}\,(1)\,(E_q) \quad \textit{imply} \ \ s_n \to s(E_p)\,.$$

(ii) For $0 < p \leq r < q \leq 1$ *we have the convexity-theorem*

$$s_n \to s(E_p) \ \ \textit{and} \quad s_n = \boldsymbol{O}\,(1)\,(E_q) \quad \textit{imply} \ \ s_n \to s(E_r)\,.$$

Obviously part (ii) is in both cases a trivial consequence of the Abelian inclusion and part (i).
The aim of this paper is to show that the above results are special cases of a more general setting.

For the following assume that (p_n) is a sequence of reals with the following properties:

$$(1.1) \qquad \begin{cases} p_0 > 0, \quad p_n \geq 0, \quad n \in I\!N, \text{ such that the power series} \\ p(t) = \displaystyle\sum_{n=0}^{\infty} p_n t^n \quad \text{has radius of convergence } \ R > 0. \end{cases}$$

Since we can use $p_n R^n$ as weights in case $0 < R < \infty$, we only have to deal with the two cases $R = 1$ and $R = \infty$.
Furthermore we define the α-th convolution $p_n^{*\alpha}$ of a sequence (p_n) by

$$p_n^{*1} := p_n, \quad n = 0, 1, 2, \ldots \quad \text{and } \ p_n^{*(\alpha+1)} := \sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k.$$

We now generalize the summability methods used in Theorems TC1 and TC2.

To this end we need a further sequence (q_n) of nonnegative reals, also satisfying (1.1), in general with a different radius of convergence R_q for the associated power series.
We then say, that a sequence (s_n) is summable to s by the

(i) **power series method** of summability (P, p), briefly $s_n \to s(P, p)$, if

$$
(1.2) \qquad \begin{cases} p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \quad \text{converges for } |t| < R \\[2ex] \text{and if} \quad \sigma_p(t) = \dfrac{p_s(t)}{p(t)} \to s \quad \text{as} \quad t \to R - . \end{cases}
$$

(In case $R = 1$ we have the so-called (J_p)-methods, in case $R = \infty$ the (B_p)-methods).

(ii) **general Nörlund-means** $(N, p^{*\alpha}, q^{*\beta})$; $\alpha, \beta \in I\!N$,
briefly $s_n \to s \ (N, p^{*\alpha}, q^{*\beta})$, if

$$
(1.3) \qquad \begin{cases} \dfrac{1}{r_n} \sum_{k=0}^{n} p_{n-k}^{*\alpha} q_k^{*\beta} s_k \to s \quad (n \to \infty), \quad \text{where we suppose that} \\[2ex] r_n := (p^{*\alpha} * q^{*\beta})_n = \sum_{k=0}^{n} p_{n-k}^{*\alpha} q_k^{*\beta} > 0 \quad \text{for} \quad n = 0, 1, \ldots . \end{cases}
$$

We require all methods to be regular. By Theorem 5 in [9], we have regularity for a power series method if and only if

$$
(1.4) \quad \begin{cases} (A) \quad P_n = \sum_{k=0}^{n} p_k \to \infty, \quad (n \to \infty), \text{ in case } R = 1, \\[2ex] (B) \quad p(t) \text{ is not a polynomial, i.e.} \\[2ex] \qquad\ \ p_n \neq 0 \text{ for infinitely many } n \text{ in case } R = \infty. \end{cases}
$$

By Theorem 3 in [9] the general Nörlund mean $(N, p^{*\alpha}, q^{*\beta})$ is regular if and only if

$$
(1.5) \qquad \frac{p_{n-k}^{*\alpha}}{r_n} \to 0 \quad \text{for any fixed } k.
$$

Remark 1: Important special cases are

(i) The Cesàro–Abel-methods:
$p_n = 1: \quad (P, p) = (A), \quad (N, p^{*\alpha}, p) = (C_\alpha), \quad \alpha \in I\!N.$

(ii) The generalized Abel-method ($\delta > 0$):
$p_n = \binom{n-1+\delta}{n}$: $(P,p) = (A_{\delta-1})$, $(N, p^{*\alpha}, \mathbf{1}) = (C_{\alpha\delta})$, $\alpha \in I\!N$.

(iii) The Euler-Borel-methods:
$p_n = 1/n!$: $(P,p) = (B)$, $(N, p^{*\alpha}, p) = (E_{\frac{1}{1+\alpha}})$, $\alpha \in I\!N$.

(We use the notation $\mathbf{1}$ for the sequence $(1, 1, \ldots)$)

We now generalize the above results to our general setting, provided some regularity assumptions are satisfied.

2 Main results

In [10], Proposition 1, R.Kiesel showed that for $\alpha \leq \beta$, $\alpha, \beta \in I\!N$ the following inclusions hold true:

$$s_n \to s(N, p^{*\alpha}, p) \Rightarrow s_n \to s(N, p^{*\beta}, p) \Rightarrow s_n \to s(P, p),$$

provided that for all $\gamma \in I\!N$ the methods $(N, p^{*\gamma}, p)$ are regular (for the second inclusion only the regularity of the (P,p)-method is needed.).

This is especially the case , if one of the following conditions is satisfied.

$$(2.1) \begin{cases} \text{(A)} \quad p_n \sim n^\sigma L(n), & \sigma \geq 0,\ n^\sigma L(n) \text{ is nondecreasing and } L(.) \text{ is slowly} \\ & \text{varying, see [3] §1.2 for the definition;} \\ \text{(B)} \quad p_n \sim \exp\{-g(n)\}, & \text{where } g \in C_2[0,\infty), \\ & \text{with } g''(x) \downarrow 0,\ x^2 g''(x) \uparrow \infty\ (x \to \infty). \end{cases}$$

Using the sequence of maximal weights" (Δ_n) defined by

$$(2.2) \qquad \Delta_n = \inf_{0<t<R} p(t) t^{-n},$$

we have in the above cases the following relationship

$$(2.3) \qquad \Delta_n = \sqrt{2\pi}\phi(n)\, p_n \quad (n \to \infty),$$

where $\phi(.)$ is a suitable, positive function.

For $(x \to \infty)$ we have in case (A) that $\sqrt{2\pi}\phi(x) \sim \Gamma(\sigma+1)\left(\frac{\sigma+1}{e}\right)^{-\sigma-1} x$ and in case (B) that $\phi(x) \sim (g''(x))^{-\frac{1}{2}}$.
Following [2,3 § 2.11] we call a function $\psi : (0,\infty) \to (0,\infty)$ self-neglecting if ψ satisfies $\psi(x) = \boldsymbol{o}\,(x)$ $(x \to \infty)$, and if $\psi(x + t\psi(x))/\psi(x) \to 1$ $(x \to \infty)$ locally uniformly in $t \in I\!R$.
Observe that $g''(x)^{-\frac{1}{2}}$ is self-neglecting because of (2.1) and since for e.g. $t \geq 0$

$$\begin{aligned}
\left(\frac{g''(x+tg''(x)^{-1/2})}{g''(x)}\right)^{-\frac{1}{2}} &\geq 1, \quad \text{and} \\
\left(\frac{g''(x+tg''(x)^{-1/2})}{g''(x)}\right)^{-\frac{1}{2}} &= \left(\frac{(x+tg''(x)^{-1/2})^2 g''(x+tg''(x)^{-1/2})}{x^2 g''(x)}\right)^{-\frac{1}{2}} \left(1+\frac{t}{\sqrt{x^2 g''(x)}}\right) \\
&\leq 1+\frac{t}{\sqrt{x^2 g''(x)}} \to 1 \;\; (x \to \infty), \;\; \text{locally uniformly in } t.
\end{aligned}$$

Because of this locally uniform convergence $\phi(.)$ is self-neglecting, too.
We can now state our main theorem

THEOREM 1
Let $\alpha, \beta, \gamma, \delta \in I\!N$ *with* $\alpha < \delta \leq \beta$ *and assume that* (p_n) *satisfies* (2.1). *Then*
(i) $s_n \to s(P, p^{*\gamma})$ *and* $s_n = \boldsymbol{O}\,(1)$ $(N, p^{*\alpha}, p^{*\gamma})$ *imply* $s_n \to s(N, p^{*\beta}, p^{*\gamma})$.
(ii) $s_n \to s(N, p^{*\beta}, p^{*\gamma})$ *and* $s_n = \boldsymbol{O}\,(1)$ $(N, p^{*\alpha}, p^{*\gamma})$ *imply* $s_n \to s(N, p^{*\delta}, p^{*\gamma})$.

Remark 2:
In case $p_n \equiv 1$, $\gamma = 1$ resp. $p_n = 1/n!$, $\gamma = 1$ Theorem 1 is Theorem TC1 resp. TC2 in the discrete index case.

In our paradigms Abel- and Borel-method we have the following relations of the methods (see [5]):
(i) Abel-case: $(A_{\alpha-1}) = (P, \binom{n+\alpha-1}{n}) = (P, \mathbf{1}^{*\alpha}), \alpha > 0$, then for $\mu > \lambda > -1$:

$$s_n \to s(A_\mu) \Rightarrow s_n \to s(A_\lambda).$$

(ii) Borel-case: Since $p_n^{*\alpha} = \alpha^n/n!$, we have

$$(B) = (P, 1/n!) \approx (P, ((\alpha^n)/n!)) = (P, (1/n!)^{*\alpha}).$$

(Where we use $\approx$ to note that two methods are equivalent.)

So the question arises what the relation of $(P, p^{*\alpha})$ and $(P, p^{*\beta})$ resp. $(N, p, p^{*\alpha})$

and $(N,p,p^{*\beta})$ in the general case is. Unfortunatly we can only present answers to the question under additional assumptions.

PROPOSITION 1

Suppose $\alpha,\beta \in I\!N$ *and the sequence* (p_n) *satisfies* (1.1) *with* $R=1$ *or* $R=\infty$ *and* $p_n^{*\alpha}>0$. *If we have furthermore that* $\mu_n=(p_n^{*\beta})/(p_n^{*\alpha})$ *is a totally monotone sequence, i.e.*

$$\mu_n = \int_0^R t^n \, d\chi(t) \ < \infty \tag{2.4}$$

for all $n=0,1,\ldots$ *with some (bounded) nondecreasing function* χ *, then we have*

$$s_n \to s(P,p^{*\alpha}) \quad \textit{implies} \quad s_n \to s(P,p^{*\beta}).$$

This result can also be obtained using a theorem of Borwein in [5], but we are able to present a somewhat easier proof.

An answer to the question of inclusion in case of the $(N,p,p^{*\alpha})$-means, was already given by Das [8], but again only under restricting additional assumptions.

PROPOSITION 2

Let $\alpha,\beta \in I\!N$ *and* (p_n) *a sequence of strictly positive reals. If*

$$\frac{p_{n+1}}{p_n} \ \uparrow 1 \quad (n\to\infty) \tag{2.5}$$

and if additionaly either

$$\frac{p_n^{*\beta}}{p_n^{*\alpha}} \geq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \textit{and } (N,p,p^{*\beta}) \textit{ is regular,}$$

or

$$\frac{p_n^{*\beta}}{p_n^{*\alpha}} \leq \frac{p_{n+1}^{*\beta}}{p_{n+1}^{*\alpha}} \quad \textit{and } \frac{p_n^{*\beta}p_n^{*\alpha+1}}{p_n^{*\beta+1}p_n^{*\alpha}} = \boldsymbol{O}(1) \textit{ and } (N,p,p^{*\alpha}) \textit{ is regular.}$$

Then $(N,p,p^{*\alpha})$ *convergence implies* $(N,p,p^{*\beta})$*-convergence.*

3 Auxiliary results

First we discuss the asymptotic properties of the (N, p, q)-means.

LEMMA 1
Assume that (p_n) *satisfies* (2.1).
(i) In case (A), *i.e.* $p_n = n^\sigma L(n)$, *we have*

$$p_n^{*2} \sim \begin{cases} n^{2\sigma+1} L^2(n) B(\sigma+1, \sigma+1), & \text{if } \sigma > -1, \\ L^*(n) n^{-1}, & \text{if } \sigma = -1, \end{cases}$$

with $B(.,.)$ *denoting the beta-integral and* $L^*(.)$ *some slowly varying function.*

(ii) In case (B), *we have for any* $\alpha \in I\!N$

$$p_n^{*\alpha} \sim \sqrt{(2\pi)^{\alpha-1}/\alpha}\ \phi(n/\alpha)^{\alpha-1} \ \exp\{-\alpha g(n/\alpha)\}\ (n \to \infty), \tag{3.1}$$

$\phi(.)$ *as in* (2.3).

Proof:
(i) are slight generalisations of Theorem 42 in [9] and Theorem 2.3.1 in chapter 5 of [20].

(ii) For $\alpha = 2$ the result is contained in Proposition 3 of [10]. We use induction on α for the general case.
By Definition we have

$$p_n^{*(\alpha+1)} = \sum_{\nu=0}^{n} p_\nu^{*\alpha} p_{n-\nu}.$$

We define a function

$$\epsilon(x) = x(x^2 g''(x))^{-1/4}. \tag{3.2}$$

Then we can show that the essential part of the sum occurs for $\nu \in M(n)$ with

$$M(n) := \left\{ \nu : \left| \nu - \frac{\alpha n}{\alpha+1} \right| \le \epsilon\left(\frac{\alpha n}{\alpha+1}\right) \right\}.$$

(Use techniques similar to those in the proof of Lemma 2 in [6], see also related calculations in [12,13] .)
By the induction hypotheses we find

$$p_n^{*(\alpha+1)} \sim \sum_{\nu \in M(n)} \sqrt{(2\pi)^{\alpha-1}/\alpha}\ \phi(\nu/\alpha)^{\alpha-1} \ \exp\{-\alpha g(\nu/\alpha)\} \ \exp\{-g(n-\nu)\}.$$

We now use the asymptotics for p_n and the Taylor-expansion ($\theta, \vartheta \in (0,1)$):

$$\begin{aligned} p_n^{*(\alpha+1)} \sim \sum_{\nu \in M(n)} & \sqrt{\frac{1}{\alpha}\left(\frac{2\pi}{g''\left(\frac{\nu}{\alpha}\right)}\right)^{\alpha-1}} \\ & \exp\left\{-\alpha\left(g\left(\frac{n}{\alpha+1}\right)+g'\left(\frac{n}{\alpha+1}\right)\left(\frac{\nu}{\alpha}-\frac{n}{\alpha+1}\right)\right.\right. \\ & \left. +\frac{1}{2}g''\left(\frac{n}{\alpha+1}+\theta\left(\frac{\nu}{\alpha}-\frac{n}{\alpha+1}\right)\right)\left(\frac{\nu}{\alpha}-\frac{n}{\alpha+1}\right)^2\right) \\ & -g\left(\frac{n}{\alpha+1}\right)-g'\left(\frac{n}{\alpha+1}\right)\left(n-\nu-\frac{n}{\alpha+1}\right) \\ & \left. -\tfrac{1}{2}g''\left(\frac{n}{\alpha+1}+\vartheta\left(n-\nu-\frac{n}{\alpha+1}\right)\right)\left(n-\nu-\frac{n}{\alpha+1}\right)^2\right\} \end{aligned}$$

Now we use the basic inequality (13) in [6], namely

$$\left|\frac{g''(t)}{g''(x)}-1\right| \le 4\frac{|t-x|}{x} \text{ for all sufficiently large } t,x, \quad \text{if } |t-x| \le x/4,$$

which is satisfied in our range $M(n)$, and the fact that $\epsilon(n)/n \to 0$ as $n \to \infty$ to obtain

$$\begin{aligned} p_n^{*(\alpha+1)} & \sim \sqrt{\frac{1}{\alpha}\left(\frac{2\pi}{g''(\frac{n}{\alpha+1})}\right)^{\alpha-1}} \times \exp\left\{-(\alpha+1)g\left(\frac{n}{\alpha+1}\right)\right\} \\ & \times \sum_{\nu \in M(n)} \exp\left\{-\frac{(\alpha+1)}{2\alpha}g''\left(\frac{n}{\alpha+1}\right)\left(\nu-\frac{n\alpha}{\alpha+1}\right)^2\right\}(1+\boldsymbol{o}(1)) \\ & \sim \sqrt{\frac{(2\pi)^\alpha}{\alpha+1}\phi\left(\frac{n}{\alpha+1}\right)^\alpha} \exp\left\{-(\alpha+1)g\left(\frac{n}{\alpha+1}\right)\right\} \end{aligned}$$

For the last step use the approximation of the sum with the integral of a Gaussian density with variance $\alpha/((\alpha+1)g''(n/(\alpha+1)))$. ■

Corollary
If (p_n) *satisfies* (2.1 B) *and* $\alpha, \beta \in I\!N$, *then we have for the entry* $a_{n,k}$ *of the*

$(N, p^{*\alpha}, p^{*\beta})$ - matrix the asymptotic relation

$$a_{n,k} \sim \sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta}}\phi\left(\frac{n}{\alpha+\beta}\right)^{-1} \exp\left\{-\frac{\alpha+\beta}{2\alpha\beta}\left(\frac{k-\frac{n\beta}{\alpha+\beta}}{\phi\left(\frac{n}{\alpha+\beta}\right)}\right)^2\right\}$$

if $\left|k - \frac{n\beta}{\alpha+\beta}\right| \leq \epsilon(n)$ *with* $\epsilon(.)$ *as in (3.2) and furthermore*

$$\sum_{\left|k-\frac{n\beta}{\alpha+\beta}\right|>\epsilon(n)} a_{n,k} \to 0 \quad (n \to \infty).$$

Proof:
If $\left|k - \frac{n\beta}{\alpha+\beta}\right| \leq \epsilon(n)$ then $(\theta, \xi \in (0,1))$

$$\frac{p^{*\alpha}_{n-k}p^{*\beta}_k}{p^{*(\alpha+\beta)}_n} \sim \frac{\exp\left\{-\alpha g\left(\frac{n-k}{\alpha}\right) - \beta g\left(\frac{k}{\beta}\right)\right\}}{\exp\left\{-(\alpha+\beta) g\left(\frac{n}{\alpha+\beta}\right)\right\}} \sqrt{\frac{(2\pi)^{\alpha-1}(2\pi)^{\beta-1} g''\left(\frac{n}{\alpha+\beta}\right)^{\alpha+\beta-1}(\alpha+\beta)}{(2\pi)^{\alpha+\beta-1} g''\left(\frac{n-k}{\alpha}\right)^{\alpha-1} g''\left(\frac{k}{\beta}\right)^{\beta-1}\alpha\beta}}$$

$$\begin{aligned}
&\sim \exp\Bigg\{ -\alpha\left(g\left(\frac{n}{\alpha+\beta}\right) + g'\left(\frac{n}{\alpha+\beta}\right)\left(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\right)\right. \\
&\left. + \frac{1}{2} g''\left(\frac{n}{\alpha+\beta} + \theta\left(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\right)\right)\left(\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\right)^2\right) \\
&- \beta\left(g\left(\frac{n}{\alpha+\beta}\right) + g'\left(\frac{n}{\alpha+\beta}\right)\left(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\right)\right. \\
&\left. + \frac{1}{2} g''\left(\frac{n}{\alpha+\beta} + \xi\left(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\right)\right)\left(\frac{k}{\beta} - \frac{n}{\alpha+\beta}\right)^2\right)\Bigg\} \\
&\times \exp\left\{(\alpha+\beta) g\left(\frac{n}{\alpha+\beta}\right)\right\} \sqrt{(1+\boldsymbol{o}(1))\frac{\alpha+\beta}{2\pi\alpha\beta} g''\left(\frac{n}{\alpha+\beta}\right)}.
\end{aligned}$$

Now $\left|\frac{k}{\beta} - \frac{n}{\alpha+\beta}\right| \leq \frac{\epsilon(n)}{\beta}$ and $\left|\frac{n-k}{\alpha} - \frac{n}{\alpha+\beta}\right| \leq \frac{\epsilon(n)}{\alpha}$. Therefore we obtain the desired result by the same calculations as used in Lemma 1.

For the second part observe that

$$\sum_{k=0}^{n} \frac{p^{*\alpha}_{n-k}p^{*\beta}_k}{p^{*(\alpha+\beta)}_n} = 1 \sim (1 + \boldsymbol{o}(1)) \sum_{\left|k-\frac{n\beta}{\alpha+\beta}\right|\leq\epsilon(n)} \exp\{\ldots\} \sqrt{\ldots} \;\blacksquare$$

We now give the asymptotics of the relevant power-series methods and show that for bounded sequences these methods are equivalent to certain generalized Valiron -type means, compare [6,11]

LEMMA 2

Assume that (p_n) *satisfies* (2.1 B). *Then we have*

(i) $\left(p\left(\exp\left\{g'\left(\frac{x}{\mu}\right)\right\}\right)\right)^{\mu} \sim \left(\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)\right)^{\mu} \exp\left\{-\left(g\left(\frac{x}{\mu}\right) - \frac{x}{\mu}g'\left(\frac{x}{\mu}\right)\right)\right\}$

for $x \to R$.

(ii) *For bounded sequences* (s_n) *the following equivalence holds true*

$$s_n \to s(P, p^{*\mu}) \quad \Leftrightarrow \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}\left(\frac{x-t}{\phi\left(\frac{x}{\mu}\right)}\right)^2\right\} s(t)\frac{dt}{\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)} \to s,$$

for $x \to \infty$, *where* $s(t) = s_{[t]}$, *for* $t \geq 0$ *and* $s(t) = 0$ *elsewhere.*

Proof:

(i) follows directly from [12], Lemma 5, resp [13], Lemma 8, see also Lemma 2 in [6]

(ii) In this case the calculations are similar to the calculations used in [6], Lemma 2 and [11],Theorem 2, so we only outline the major steps. We have by using Lemma 1 and part (i) (For the notation see (1.2)).

$$\begin{aligned}
\sigma_{p^{*\mu}}\left(e^{g'\left(\frac{x}{\mu}\right)}\right) &= \frac{(1+\boldsymbol{o}(1))}{\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)}\sum_{n=0}^{\infty} s_n \exp\left\{-\mu g\left(\frac{n}{\mu}\right) + ng'\left(\frac{x}{\mu}\right) + \mu g\left(\frac{x}{\mu}\right) - xg'\left(\frac{x}{\mu}\right)\right\} \\
&= \frac{(1+\boldsymbol{o}(1))}{\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)}\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\mu}{2}g''\left(\frac{x}{\mu} + \theta\left(\frac{n}{\mu} - \frac{x}{\mu}\right)\right)\left(\frac{n}{\mu} - \frac{x}{\mu}\right)^2\right\} \\
&= \frac{(1+\boldsymbol{o}(1))}{\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)}\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{1}{2\mu}g''\left(\frac{x}{\mu}\right)\left(\frac{n}{\mu} - \frac{x}{\mu}\right)^2\right\} \\
&= (1+\boldsymbol{o}(1))\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}\left(\frac{x-t}{\phi\left(\frac{x}{\mu}\right)}\right)^2\right\} s(t)\frac{dt}{\sqrt{2\pi\mu}\phi\left(\frac{x}{\mu}\right)} \quad \blacksquare
\end{aligned}$$

Next we show that the $(N, p^{*\alpha}, p^{*\beta})$-means generalize some important properties of the Euler means.

First we consider the well known product-formula for the Euler-means

$$E_\alpha \circ E_\beta = E_{\alpha+\beta}.$$

This becomes

LEMMA 3

Assume that (p_n) *and* (q_n) *satisfy* (1.1) *(with possibly different radii of convergence) and let* $\alpha, \beta, \gamma \in I\!N, \alpha \le \beta$.

(i) With $r^{*(\alpha+\beta)} := p^{*\alpha} * q^{*\beta}$ *, we have*

$$(N, p^{*\beta}, q^{*\gamma}) = (N, p^{*(\beta-\alpha)}, r^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, q^{*\gamma}) \tag{3.3}$$

resp. in case $(p_n) = (q_n)$

$$(N, p^{*\beta}, p^{*\gamma}) = (N, p^{*(\beta-\alpha)}, p^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, p^{*\gamma}).$$

(ii) If $(N, p^{*(\beta-\alpha)}, r^{*(\alpha+\gamma)})$ *is regular, then*

$s_n \to s(N, p^{*\alpha}, q^{*\gamma})$ *implies* $s_n \to s(N, p^{*\beta}, q^{*\gamma})$.

Proof:
(ii) is a trivial consequence of (i).
To prove (i) observe that

$$(p^{*(\beta-\alpha)} * r^{*(\alpha+\gamma)})_n = (p^{*\beta} * q^{*\gamma})_n$$

and

$$\sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{k-\nu}^{*\alpha} q_\nu^{*\gamma} s_\nu$$

$$= \sum_{\nu=0}^{n} q_\nu^{*\gamma} s_\nu \sum_{k=0}^{n-\nu} p_{n-\nu-k}^{*(\beta-\alpha)} p_k^{*\alpha} = \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_\nu^{*\gamma} s_\nu .$$

Now $s_n \to s\ (N, p^{*(\beta-\alpha)}, r^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, q^{*\gamma})$ means that

$$\frac{1}{(p^{*(\beta-\alpha)} * r^{*(\alpha+\gamma)})_n} \sum_{k=0}^{n} p_{n-k}^{*(\beta-\alpha)} r_k^{*(\alpha+\gamma)} \frac{1}{r_k^{*(\alpha+\gamma)}} \sum_{\nu=0}^{k} p_{k-\nu}^{*\alpha} q_\nu^{*\gamma} s_\nu \to s \quad (n \to \infty),$$

but by the above identities this is the same as

$$\frac{1}{(p^{*\beta} * q^{*\gamma})_n} \sum_{\nu=0}^{n} p_{n-\nu}^{*\beta} q_\nu^{*\gamma} s_\nu \to s \quad (n \to \infty),$$

which is $(N, p^{*\beta}, q^{*\gamma})$ convergence. ■

A classical result of Knopp [9, Theorem 149] gives a connection between Cesáro convergence with speed and Euler convergence. We generalize this for general (p_n) with an additional condition on the sequence (s_n). (In [10, Theorem 2] this generalization is given with an additional condition on the (p_n), but without conditions on the (s_n).)

LEMMA 4
Let (p_n) be a sequence of weights satisfying (2.1B) *and $\phi(.)$ as in* (2.3). *Furthermore assume that $s_n = \boldsymbol{O}(1)$. Then*

$$\frac{1}{n+1}\sum_{k=0}^{n}(s_k+\epsilon_k) = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right), \quad (n \to \infty), \quad \textit{with some nullsequence}\,(\epsilon_n)$$

implies $s_n \to s(N, p^{*\alpha}, p^{*\beta})$ for every $\alpha, \beta \in I\!N$.

Proof:
Since $s_n = \boldsymbol{O}(1)$ we can use the asymptotic weights computed in the Corollary to Lemma 1 in the $(N, p^{*\alpha}, p^{*\beta})$ method. By inclusion we have only to show the implication for the $(N, p, p^{*\beta})$ method. Because of regularity and linearity we can suppose $s = 0$ and omit the convergent sequence (ϵ_k). Thus the hypothesis becomes

$$\sum_{k=0}^{n} s_k = \boldsymbol{o}(\phi(n)) \;\; (n \to \infty).$$

For given $\epsilon > 0$ we can find a $N \in I\!N$ such that for $n \geq l \geq m \geq N$

$$\left|\sum_{k=m}^{l} s_k\right| \leq \epsilon\phi(l) \leq \epsilon\phi(n),$$

using also the monotonicity of $\phi(.)$. By the Corollary to Lemma 1 and since $s_n = \boldsymbol{O}(1)$ we have for the $(N, p, p^{*\beta})$-transform t_n

$$t_n = \sqrt{\frac{\beta+1}{2\pi\beta}}\phi\left(\frac{n}{\beta+1}\right)^{-1} \sum_{\left|k-\frac{n\beta}{\beta+1}\right| \leq \epsilon(n)} \exp\left\{-\frac{\beta+1}{2\beta}\left(\frac{k-\frac{n\beta}{\beta+1}}{\phi\left(\frac{n}{\beta+1}\right)}\right)^2\right\} s_k \;+ \boldsymbol{o}(1),$$

with a function $\epsilon(.)$ as in (3.2). So the weights are piecewise monotonic and the maximal weight is for $k = \frac{n\beta}{\beta+1}$. We therefore split the sum in two parts, namely

$$t_n = \sum_{\frac{n\beta}{\beta+1}-\epsilon(n) \leq k < \frac{n\beta}{\beta+1}} \ldots + \sum_{\frac{n\beta}{\beta+1} \leq k \leq \frac{n\beta}{\beta+1}+\epsilon(n)} \ldots + \boldsymbol{o}(1).$$

Using Abels partial summantion and the monotonicity of the weights we find that each of the two sums is bounded by $\epsilon \frac{\phi(n)}{\phi(n/\beta+1)}$. Since $\phi(n/\gamma) = \boldsymbol{O}\,(\phi(n))$ for any fixed $\gamma > 0$, we obtain the desired result. ■

Cesàro-convergence with speed is also connected to the methods of moving-averages by the following

PROPOSITION 3

The following statements are equivalent for a self-neglecting function $\phi(.)$

(i) $\displaystyle \frac{1}{n+1}\sum_{k=0}^{n}(s_k+\epsilon_k) = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right) \quad (n\to\infty)$ *for some* $\epsilon_n \to 0$.

(ii) $\displaystyle \frac{1}{u\phi(n)}\sum_{n\le k<n+u\phi(n)} s_k \to s, \quad \forall u>0, \quad (n\to\infty).$

For the proof see [2], for notation and properties of self-neglecting functions consult [3, §2.11].

In the Euler–Borel case we have the identity $(B)\circ(E_p) \approx (B)$. A similar identity can be obtained in the general case. For a related calculation compare [7].

LEMMA 5

Assume that (p_n) *and* (q_n) *satisfy* (1.1) *with the same radius of convergence* R *and let* $\alpha, \beta \in I\!N$ *then*

$$s_n \to s(P, q^{*\beta}) \quad \Leftrightarrow s_n \to s(P, r^{*(\alpha+\beta)}) \circ (N, p^{*\alpha}, q^{*\beta}).$$

Proof:

$s_n \to s(P, q^{*\beta})$ means that $\dfrac{\sum_{n=0}^{\infty} s_n q_n^{*\beta} x^n}{(q(x))^{\beta}} \to s \;\; (x\to R)$,

and $s_n \to s(P, r^{*(\alpha+\beta)}) \circ (N, p^{*\alpha}, q^{*\beta})$ means that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}^{*\alpha} s_k q_k^{*\beta}\right) x^n}{(p(x))^{\alpha}(q(x))^{\beta}} \to s \;\; (x\to R).$$

But

$$\frac{\sum_{n=0}^{\infty} s_n q_n^{*\beta} x^n}{(q(x))^\beta} = \frac{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_{n-k}^{*\alpha} s_k q_k^{*\beta}\right) x^n}{(p(x))^\alpha (q(x))^\beta},$$

and this proves the proposition. ■

Using Borweins Theorem, i.e. Proposition 1, we obtain

Corollary

If the assumptions of Lemma 5 hold true and if $\dfrac{q_n^{*\beta}}{r_n^{*(\alpha+\beta)}}$ *is a totally monotone sequence, then*

$$s_n \to s\ (P, q^{*\beta}) \quad \Rightarrow \quad s_n \to s\ (P, q^{*\beta}) \circ (N, p^{*\alpha}, q^{*\beta}).$$

Generalizing Theorem 1 in [10] slightly we obtain the following Tauberian theorem

THEOREM 2

Assume that (p_n) *satisfies* (1.1) *and* (2.1 B). *Then we have under the Tauberian condition* $s_n = \boldsymbol{O}(1)$ *that for any* $\gamma \in \mathbb{N}$

$$s_n \to s(P, p^{*\gamma}) \quad \text{implies } s_n \to s(N, p^{*\alpha}, p^{*\beta})$$

for all $\alpha, \beta \in \mathbb{N}$.

Remark 3:

(i) Under (2.1) $(N, p^{*\alpha}, p^{*\beta})$ is regular for all $\alpha, \beta \in \mathbb{N}$.

(ii) $s_n \to s(N, p^{*\alpha}, p^{*\beta})$ implies always $s_n \to s(P, p^{*\beta})$, since

$$\sigma_{p^{*\beta}}(t) = \frac{\sum_{n=0}^{\infty} s_n p_n^{*\beta} x^n}{(p(x))^\beta} = \frac{\sum_{n=0}^{\infty} p_n^{*(\alpha+\beta)} \frac{1}{p_n^{*(\alpha+\beta)}}\left(\sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k^{*\beta} s_k\right) x^n}{(p(x))^\alpha (p(x))^\beta},$$

and since $(P, p^{*(\alpha+\beta)})$ is regular, the Abelian conclusion follows.

Proof:

By Lemma 3 (ii), it is sufficient to consider $\alpha = 1$.
Define $s(u) = s_{[u]}$ if $u \geq 0$ and $s(u) = 0$ if $u < 0$ and $K(x) = 1/\sqrt{2\pi}\exp\{-x^2/2\}$.

Since $s_n = \boldsymbol{O}(1)$ we have by Lemma 2 (ii), that $s_n \to s(P, p^{*\gamma})$ implies

$$(3.4) \qquad \lim_{x\to\infty} \int_{-\infty}^{\infty} K\left(\frac{x-t}{\sqrt{\gamma}\phi\left(\frac{x}{\gamma}\right)}\right) s(t) \frac{dt}{\sqrt{\gamma}\phi(x/\gamma)} = s$$

The conditions of Theorem 1 of [15], i.e. $K(x) \in L^1(-\infty, \infty)$, the Fourier- transform of K is nonvanishing for any real argument and $\phi(.)$ is self-neglecting, are trivially satisfied.
It follows now from that theorem that if we choose $\epsilon > 0$ and define

$$H(x) = \begin{cases} \frac{1}{\epsilon}, & \text{if} \quad x \in (-\epsilon, 0), \\ 0, & \quad\quad x \notin (-\epsilon, 0), \end{cases}$$

that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} H\left(\frac{x-t}{\sqrt{\gamma}\phi(x)}\right) s(t) \frac{dt}{\sqrt{\gamma}\phi(x/\gamma)} = \lim_{x\to\infty} \frac{1}{\epsilon\sqrt{\gamma}\phi(x/\gamma)} \sum_{x \le k < x+\epsilon\sqrt{\gamma}\phi(x/\gamma)} s_k = s.$$

Because $\phi(.)$ is self-neglecting and $\phi(x/\gamma) = \boldsymbol{O}(\phi(x))$, for any fixed $\gamma > 0$, we obtain by Proposition 3 , that

$$\frac{1}{n+1} \sum_{k=0}^{n} (s_k + \epsilon_k) = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right),$$

which in turn by Lemma 4 implies that $s_n \to s(N, p, p^{*\beta})$.■

4 Proofs

Proof of Theorem 1:
Part (i): By Lemma 5:

$$s_n \to s(P, p^{*\gamma}) \Leftrightarrow s_n \to s(P, p^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, p^{*\gamma}).$$

In case (A)
We apply Karamatas Tauberian theorem (observe Lemma 1) (see [2, Theorem 1.7.6 , 18]) and obtain

$$s_n \to s(N, \mathbf{1}, p^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, p^{*\gamma}).$$

Since $s_n = \boldsymbol{O}(1)\,(N, p^{*\alpha}, p^{*\gamma})$ we can use the asymptotic weights and assume w.l.o.g that $p_n^{*(\beta-\alpha)}$ is nondecreasing and by Theorem 3 in Das [8] we get

$$s_n \to s(N, p^{*(\beta-\alpha)}, p^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, p^{*\gamma}),$$

which by Lemma 3(i) implies our result.
In case (B)
Since $s_n = \boldsymbol{O}(1)\,(N, p^{*\alpha}, p^{*\gamma})$ we can use Theorem 2 to obtain directly

$$s_n \to s(N, p^{*(\beta-\alpha)}, p^{*(\alpha+\gamma)}) \circ (N, p^{*\alpha}, p^{*\gamma}).$$

The last step is as above.

Part (ii) is directly implied by part (i) by the Abelian inclusion. ■

Remark 4:
Boos/Tietz [4] gave an alternative proof of Theorem 1 in the Borel-case. The basic steps are as follows ($\alpha = \gamma = 1, \beta = 2$)

(i) $s_n \to s(P, p) \Rightarrow s_n \to s(P, p^{*3})(N, p^{*2}, p)$

(ii) $(N, p^{*2}, p) = (N, p, p^{*2})(N, p, p)$.
Hence if $(*)$ $((N, p, p^{*2})x)_n - ((N, p, p^{*2})x)_{n-1} = \boldsymbol{O}(1/\phi(n))$ for bounded sequences (x_n) , one can use the **O**-Tauberian theorems in [12,13] to conclude

(iii) $s_n \to s(N, p^{*2}, p)$.

The statement $(*)$ in (ii) is true for some special cases, like $p_n = 1/n!$, but has not been obtained in general so far.

Proof of Proposition 1:
Observe that e.g. in case $R = \infty$

$$\sigma_{p^{*\beta}}(x) = \frac{\sum_{n=0}^{\infty} s_n \frac{p_n^{*\beta}}{p_n^{*\alpha}} p_n^{*\alpha} x^n}{(p(x))^\beta} = \int_0^\infty \frac{p(xt)^\alpha}{p(x)^\beta} \sigma_{p^{*\alpha}}(xt)\, d\chi(t) = L\left(\sigma_{p^{*\alpha}}(.), x\right).$$

The interchange of integral and sum is allowed because of the absolute convergence for $x > 0$. We now follow the arguments in an unpublished paper by A. Jakimovski (oral communication, see also [16] for details.)
L(f,x) is a positive linear operator on a linear space of real functions in $C[0, \infty)$ with the properties:

(i) There exists $e(t) > 0, e(t) \to 1, t \to \infty$ such that $L(e(.), x) \to 1, x \to \infty$, namely $e(t) = \sigma_{p^{*\alpha}}(t)$ with the sequence (s_n) chosen to be $(1, 1, \ldots)$.

(ii) There exists some $e_0(t) > 0$ such that $L(e_0(.), x) \to 0,\ x \to \infty$, namely $e_0(t) = \sigma_{p^{*\alpha}}(t) = p_0^{*\alpha}/p(t)^\alpha$, with the sequence (s_n) chosen to be $(1, 0, 0, \ldots)$.

From (i) and the assumptions we find

$$|f(t) - se(t)| < \epsilon/2 \leq \epsilon e(t), \quad \text{for } \ t \geq t_0(\epsilon),$$

and by (ii)

$$|f(t) - se(t)| \leq M \leq \frac{M}{m} e_0(t), \quad t \in [0, t_0(\epsilon)],$$

with suitable M, m. Hence for $t \geq 0$:

$$|f(t) - se(t)| \leq \epsilon e(t) + \frac{M}{m} e_0(t).$$

Since L is linear and positive we obtain that $L(f(.), x) \to s$ if $f(x) \to s$, which yields the desired result. ■

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