The Law of the Iterated logarithm for certain Power Series and generalized Nörlund methods[1]

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1 *Introduction and main result*

Let (Ω, Σ, P) be a probability space and suppose all random variables are defined on this space.

Let $X, X_0, X_1, X_2, \ldots$ be (without further comment) throughout independent, identically distributed (i.i.d.) random variables. Lai [16] discussed the rate at which convergence takes place in the strong laws of large numbers for the classical scales of summability methods Cesàro (C_β) – Abel (A) (the discounting method) and Euler (E_γ) – Borel (B).

Theorem Lai A *The following are equivalent:*

(i) $E(X) = 0,\ E(X^2) = 1$ *;*

(ii) $$\limsup_{n\to\infty} \frac{\left|\sum_{k=0}^{n} \binom{n-k+\beta-1}{n-k} X_k\right|}{(2n^{2\beta-1}\log\log n)^{\frac{1}{2}}} = \frac{1}{(2\beta-1)^{\frac{1}{2}}\Gamma(\beta)^{\frac{1}{2}}} \quad \textit{a.s. for } \beta \geq 1\,;$$

(iii) $$\limsup_{t\to 1-} \frac{\left|\sum_{k=0}^{\infty} t^k X_k\right|}{(\frac{1}{1-t}\log|\log(1-t^2)|)^{\frac{1}{2}}} = 1 \quad \textit{a.s.}.$$

Theorem Lai B *The following are equivalent:*

(i) $E(X) = 0,\ E(X^2) = 1,\ E(X^4(\log^+|X|+1)^{-2})$ *;*

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$$\text{(ii)}\qquad \limsup_{n\to\infty} \frac{(\pi\gamma n)^{\frac{1}{4}}\dfrac{1}{\gamma+1}\left|\sum_{k=0}^{n}\binom{n}{k}\gamma^{n-k}X_k\right|}{\sqrt{\frac{1}{2}(\gamma+1)\log n}} = 1 \quad a.s.\ for\ \gamma>0\,;$$

$$\text{(iii)}\qquad \limsup_{t\to\infty} \frac{(4\pi t)^{\frac{1}{4}}e^{-t}\left|\sum_{k=0}^{\infty}\dfrac{t^k}{k!}X_k\right|}{\sqrt{\log t}} = 1 \quad a.s..$$

So we observe that in the Cesàro–Abel case we have a behaviour of *iterated logarithm* type, whereas the Euler–Borel case exhibits a *single logarithm* type behaviour.

In [13, 14] the author obtained strong laws for general classes of summability methods containing the above methods as special cases. Therefore the questions arise, what the rate of convergence in these strong laws is and where the switch from iterated to single logarithm behaviour takes place. The aim of this note is to provide an answer to the first question.

Let us first define two general classes of summability methods.

We assume that (p_n) is a sequence of reals with

$$\text{(1.1)}\qquad \begin{cases} p_0>0, \quad p_n\ge 0, \quad n\in I\!N, \text{ such that the power series} \\ p(t)=\sum_{n=0}^{\infty}p_nt^n \quad \text{has radius of convergence } R=1. \end{cases}$$

For $\beta>0$ we define the sequence (p_n^β) by the convolution

$$\text{(1.2)}\qquad p_n^\beta := \sum_{k=0}^{n}\binom{n-k+\beta-1}{n-k}p_k;\ n=0,1,\ldots$$

We apply the following methods of summability to sequences (s_n) of complex numbers:

(a) the *generalized Nörlund methods* $(N_{\beta p})$, defined by the transform

$$\text{(1.3)}\qquad \sigma_n^\beta := \frac{1}{p_n^\beta}\sum_{k=0}^{n}\binom{n-k+\beta-1}{n-k}p_ks_k,$$

and we say $s_n \to s(N_{\beta p})$ if $\sigma_n^\beta \to s\ (n \to \infty)$;
(b) the *power series method* (J_p), where we say

$$(1.4) \qquad \begin{cases} s_n \to s(J_p), & \text{if } p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \quad \text{converges for } |t| < 1 \\ & \text{and if } \quad \sigma_p(t) = \dfrac{p_s(t)}{p(t)} \to s \text{ as } \quad t \to 1- . \end{cases}$$

Observe that the Cesàro (C_β) - and the Abel (A) - method are the special cases with $p_n = 1, n = 0, 1, \ldots$. For further properties of these summability methods see [11, 14] and the references therein.
In order to prevent the weights p_n from being too irregular, we use the following regularity condition:

$$(1.6) \qquad p_n = n^\alpha L(n) \quad (n \to \infty)$$

with the parameter $\alpha > -\frac{1}{2}$ and the function $L(.)$ slowly varying,
i.e. $\frac{L(\lambda n)}{L(n)} \to 1\ (n \to \infty)\ \forall \lambda > 0$, compare [3, 17] for further details.

We now state our main results

THEOREM 1 *Let p_n be a sequence satisfying* (1.6) *and denote by $B(.,.)$ the Beta-function. The following statements are equivalent:*

(i) $E(X) = 0,\ E(X^2) = 1$ *;*

(ii) $$\limsup_{n\to\infty} \frac{\left| \sum_{k=0}^{n} \binom{n-k+\beta-1}{n-k} p_k X_k \right|}{\left(2B(2\alpha+1, 2\beta-1) n^{2(\alpha+\beta)-1} L(n) \log\log n\right)^{\frac{1}{2}}} = 1 \quad a.s \text{ for } \beta \geq 1 \,;$$

(iii) $$\limsup_{t\to 1-} \frac{\left| \sum_{k=0}^{\infty} p_k t^k X_k \right|}{\left(2 \sum_{k=0}^{\infty} p_k^2 t^{2k} \log \left| \log \left(\sum_{k=0}^{\infty} p_k^2 t^{2k} \right) \right| \right)^{\frac{1}{2}}} = 1 \quad a.s..$$

We see that, for polynomial-like growth of the weight sequence (p_n) we have an iterated logarithm type of behaviour. Our conjecture is, that for exponentially growing weight sequences like $p_n = \exp\{n^\alpha\}, 0 < \alpha < 1$ we have a behaviour of single logarithm type (see [12] for some results hinting in this direction) and hence the switch occurs in the intermediate grey zone.

Under the assumptions of Theorem 1 we also can identify the limit points.

THEOREM 2 *Under the above assumptions the cluster set in (ii) and (iii) above is* $[-1,1]$ *a.s.*.

REMARK: That the cluster set is $[-1,1]$ a.s. in case of the Abel-method was quite recently shown by Bovier, Picco [4] and also by Horvàth [10], who used a different method.

In what follows we will focus on proving Theorem 1 and Theorem 2 for the power series case. We start in §2 by establishing an upper bound for the a.s. $\limsup$ behaviour, see Proposition 1. In §3 we show by modifying an argument of Feller [8, 2, 19] that a.s. boundedness of the $\limsup$ already implies the moment conditions of Theorem 1(i), see Proposition 2. In §4 we use de Acosta's [1, 2] techniques to establish Theorem 2 and hence that the $\limsup$ in Theorem 1(iii) is 1 a.s., see Proposition 3. A main tool in doing this is the Central Limit Theorem for summability methods given by Embrechts and Maejima [7]. In §5 we briefly outline how to modify the arguments to prove the Law of Iterated logarithm for the corresponding generalized Nörlund transforms. We close by supplying an appendix on the asymptotic behaviour of sums and convolutions of weight sequences satisfying (1.6), not necessarily assuming that $\alpha > -1/2$.

2 *The* $\limsup$ *-behavior*

We start with some notation:

$$p_X(t) := \sum_{k=0}^{\infty} p_k X_k e^{-\frac{k}{t}} \qquad P_n := \sum_{k=0}^{n} p_k \qquad P(t) := \sum_{k\le t} p_k;$$

$$V(t) := \sum_{k=0}^{\infty} p_k^2 e^{-\frac{2k}{t}} \qquad P_n^{(2)} := \sum_{k=0}^{n} p_k^2 \qquad P^{(2)}(t) := \sum_{k\le t} p_k^2;$$

$$\varphi(t) := (2V(t)\,\mathrm{loglog}V(t))^{\frac{1}{2}}, \quad N(t) := [t\,\mathrm{log}t].$$

Observe that we used a parameter transformation. This transformation has been proven useful in many problems concerning power series methods, see [11, 12, 15] and the references cited there. We use $c(\alpha)$ as a general positive constant only depending on α. We also introduce a sequence

$$t_n := \exp\left\{\frac{n}{(\log n)^{\alpha+2}}\right\}, \tag{2.1}$$

where the parameter α depends on the weights according to (1.6).

We start by showing that the upper tail of the series is neglectible

LEMMA 1 *For a sequence* (p_n) *satisfying* (1.6) *we have*

$$\lim_{t\to\infty} \varphi(t)^{-1}\left|\sum_{k=N(t)}^{\infty} p_k X_k e^{-\frac{k}{t}}\right| = 0 \quad a.s.. \tag{2.2}$$

Proof: By the Borel-Cantelli Lemma (2.2) will follow, if we prove that for every $\epsilon > 0$

$$\sum_{n=n_0}^{\infty} P\left(\sup_{t_n\le t\le t_{n+1}}\left|\sum_{k=N(t)}^{\infty} p_k X_k e^{-\frac{k}{t}}\right| \ge \epsilon\varphi(t_n)\right) < \infty. \tag{2.3}$$

We have

$$\begin{aligned}
\sup_{t_n\le t\le t_{n+1}}\left|\sum_{k=N(t)}^{\infty} p_k X_k e^{-\frac{k}{t}}\right| &\le \sum_{k=N(t_n)}^{\infty} p_k e^{-\frac{k}{t_{n+1}}}|X_k| \\
&= \sum_{k=N(t_n)}^{\infty} p_k e^{-\frac{k}{t_{n+1}}}\{|X_k| - E(|X_k|)\} \\
&+ \sum_{k=N(t_n)}^{\infty} p_k e^{-\frac{k}{t_{n+1}}} E(|X_k|).
\end{aligned}$$

Set $N := N(t_n)$. Using the asymptotic results of Lemmas A1–A3, we get for n large enough and some constant $c(\alpha)$

$$\varphi(t_n)^{-\frac{1}{2}}\sum_{k=N}^{\infty} p_k e^{-\frac{k}{t_{n+1}}} E(|X|) \le c(\alpha)E(|X|)\frac{(t_{n+1}\log t_{n+1})^{\alpha+1}L(t_{n+1}\log t_{n+1})}{t_n^{\alpha+\frac{1}{2}}L(t_n)(\log\log t_n)^{\frac{1}{2}}t_n^{\frac{3}{4}}} = \boldsymbol{o}(1).$$

By Chebyshev's inequality (and using the lemmas A1-A3 successivly) we get for the first series

$$
\begin{aligned}
& P\left(\sum_{k=N}^{\infty} p_k e^{-\frac{k}{t_{n+1}}}\left\{|X_k| - E(|X_k|)\right\} \geq \epsilon\varphi(t_n)\right) \\
\leq\ & \frac{\sum\limits_{k=N}^{\infty} p_k^2 e^{-\frac{2k}{t_{n+1}}} Var\left(|X_k| - E(|X_k|)\right)}{\epsilon^2 \varphi(t_n)^2} \\
\leq\ & c(\alpha)\frac{(\log t_n)^{2\alpha+1}}{(\log\log t_n) t_n^{\frac{3}{2}}}.
\end{aligned}
$$

Since this is the general term of a summable sequence, the Lemma is proved. ■

We now compare the behavior of the series with parameter values between t_n and t_{n+1}.

LEMMA 2 *For a sequence* (p_n) *satisfying* (1.6) *we have*

$$
\lim_{t\to\infty} \sup_{t_n \leq t \leq t_{n+1}} \left| \frac{p_X(t)}{\varphi(t)} - \frac{p_X(t_n)}{\varphi(t_n)} \right| = 0 \quad a.s.. \tag{2.4}
$$

Proof: Since $N(t_{n+1})/N(t_n) \to 1$ $(n \to \infty)$ we can (compare the proof of Lemma 1) use $N := N(t_n)$ as upper index in the range $t_n \leq t \leq t_{n+1}$. Put $S_n = \sum\limits_{k=0}^{n} p_k X_k$; then we get by Abel's partial summation

$$
\begin{aligned}
& \sum_{k=0}^{N} p_k X_k \left(\frac{e^{-\frac{k}{t}}}{\varphi(t)} - \frac{e^{-\frac{k}{t_n}}}{\varphi(t_n)}\right) \\
= \quad & \sum_{k=0}^{N} S_k \left(\frac{e^{-\frac{k}{t}} - e^{-\frac{k+1}{t}}}{\varphi(t)} - \frac{e^{-\frac{k}{t_n}} - e^{-\frac{k+1}{t_n}}}{\varphi(t_n)}\right) + S_{N+1}\left(\frac{e^{-\frac{N+1}{t}}}{\varphi(t)} - \frac{e^{-\frac{N+1}{t_n}}}{\varphi(t_n)}\right) \\
=: \quad & I + II.
\end{aligned}
$$

Consider the *sup* of the second part first.

$$\begin{aligned}
II &\leq \sup_{t_n<t\leq t_{n+1}} |S_{N+1}| \left| \frac{e^{-\frac{N+1}{t}}}{\varphi(t)} - \frac{e^{-\frac{N+1}{t_n}}}{\varphi(t_n)} \right| \\
&\leq 2 \left(\frac{|S_{N+1}|}{(2P^{(2)}_{N+1} \operatorname{loglog} P^{(2)}_{N+1})^{\frac{1}{2}}} \right) \left(\frac{(2P^{(2)}_{N+1} \operatorname{loglog} P^{(2)}_{N+1})^{\frac{1}{2}}}{\varphi(t_n)} e^{-\frac{N+1}{t_{n+1}}} \right).
\end{aligned}$$

The first factor is bounded by the Chow-Teicher [5] LIL for weighted means. By Lemma A1 we see that the quotient of the second factor behaves like a slowly varying function. By Lemma A3 $e^{-\frac{N+1}{t_{n+1}}} \leq 1/\sqrt{t_{n+1}}$. Hence we have $II = \boldsymbol{o}(1) \quad (n \to \infty)$. To estimate the first part we start by using the triangle inequality and get

$$\begin{aligned}
I &= \left| \left(\frac{1-e^{-\frac{1}{t}}}{\varphi(t)} \right) \sum_{k=0}^{N} S_k e^{-\frac{k}{t}} - \left(\frac{1-e^{-\frac{1}{t_n}}}{\varphi(t_n)} \right) \sum_{k=0}^{N} S_k e^{-\frac{k}{t}} \right. \\
&+ \left. \left(\frac{1-e^{-\frac{1}{t_n}}}{\varphi(t_n)} \right) \sum_{k=0}^{N} S_k e^{-\frac{k}{t}} - \left(\frac{1-e^{-\frac{1}{t_n}}}{\varphi(t_n)} \right) \sum_{k=0}^{N} S_k e^{-\frac{k}{t_n}} \right| \\
&\leq \left| \left(\frac{(1-e^{-\frac{1}{t}})}{\varphi(t)} - \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \right) \sum_{k=0}^{N} S_k e^{-\frac{k}{t}} \right| \\
&+ \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \sum_{k=0}^{N} |S_k| \left(e^{-\frac{k}{t}} - e^{-\frac{k}{t_n}} \right).
\end{aligned}$$

Taking the sup over the range $t_n \leq t \leq t_{n+1}$ we get

$$\begin{aligned}
I &\leq \sup_{t_n<t\leq t_{n+1}} \left| \frac{(1-e^{-\frac{1}{t}})}{\varphi(t)} - \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \right| \sum_{k=0}^{N} |S_k| e^{-\frac{k}{t_{n+1}}} \\
&+ \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \sum_{k=0}^{N} |S_k| \left(e^{-\frac{k}{t_{n+1}}} - e^{-\frac{k}{t_n}} \right) := I_1 + I_2.
\end{aligned}$$

Now using the elementary inequality $|1-e^x| \leq |x|$ for x close to 0 and Lemma A3, we get for $t_n \leq t \leq t_{n+1}$

$$\left| \frac{1-e^{-\frac{1}{t}}}{\varphi(t)} - \frac{1-e^{-\frac{1}{t_n}}}{\varphi(t_n)} \right| \leq \frac{\left(1-\frac{\varphi(t_n)}{\varphi(t_{n+1})}\right) + \left(1-\frac{t_n}{t_{n+1}}\right)}{\varphi(t_n)t_n} \leq \frac{\alpha+2}{\varphi(t_n)t_n(\log n)^{\alpha+2}}.$$

To estimate $|S_n|$ we use the Chow-Teicher LIL for weighted means, see [5]. We start by splitting the sum in I_1 in two parts and set

$N_2 := N_2(t_n) := 2(\alpha+2)t_n \operatorname{loglog} t_n$. Using Lemmas A1 and A3 we get for n large enough

$$\begin{aligned}\sum_{k=N_2}^{N} |S_k| e^{-\frac{k}{t_{n+1}}} &\le e^{-\frac{N_2}{t_{n+1}}} \sum_{k=N_2}^{N} |S_k| \le 2e^{-\frac{N_2}{t_{n+1}}} N(P_{N+1}^{(2)} \operatorname{loglog}(P_{N+1}^{(2)}))^{\frac{1}{2}} \\ &\le 2c(\alpha)(\log t_n)^{-(\alpha+2)}(t_n \log t_n)^{\alpha+\frac{3}{2}} L(t_n \log t_n) \log\log(t_n \log t_n).\end{aligned}$$

Combining the last two inequalities we get

$$\begin{aligned}&\sup_{t_n<t\le t_{n+1}} \left| \frac{(1-e^{-\frac{1}{t}})}{\varphi(t)} - \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \right| \sum_{k=N_2}^{N} |S_k| e^{-\frac{k}{t_{n+1}}} \\ \le\ & c(\alpha) \frac{(t_n \log t_n)^{\alpha+\frac{3}{2}} L(t_n \log t_n) \log\log(t_n \log t_n)}{(\log t_n)^{\alpha+2} t_n (t_n)^{\alpha+\frac{1}{2}} L(t_n) \log\log(t_n)} \to 0.\end{aligned}$$

For the first part of the sum in I_1 we get the similar bound

$$\begin{aligned}\sum_{k=0}^{N_2} |S_k| e^{-\frac{k}{t_{n+1}}} &\le \sum_{k=0}^{N_2} |S_k| \le 2N_2(2P_{N_2}^{(2)} \operatorname{loglog}(N_2^2 P_{N_2}^{(2)}))^{\frac{1}{2}} \\ &\le c(\alpha)(t_n \log\log t_n)^{\alpha+\frac{3}{2}} L(t_n \log\log t_n)(\log\log(t_n \log\log t_n))^{\frac{1}{2}}.\end{aligned}$$

With these estimates we get

$$\begin{aligned}&\sup_{t_n<t\le t_{n+1}} \left| \frac{(1-e^{-\frac{1}{t}})}{\varphi(t)} - \frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)} \right| \sum_{k=0}^{N_2} |S_k| e^{-\frac{k}{t_{n+1}}} \\ \le\ & c(\alpha) \frac{(t_n \log\log t_n)^{\alpha+\frac{3}{2}} L(t_n \log\log t_n)(\log\log(t_n \log\log t_n))^{\frac{1}{2}}}{t_n^{\alpha+\frac{3}{2}} L(t_n)(\log\log t_n)^{\frac{1}{2}} (\log n)^{(\alpha+2)}} \\ \le\ & c(\alpha) \frac{(\log(n/(\log n)^{\alpha+2})^{\alpha+\frac{3}{2}}}{(\log n)^{\alpha+2}} \to 0 \quad (n \to \infty).\end{aligned}$$

So it remains to estimate I_2. We have

$$\begin{aligned}
&\frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)}\sum_{k=0}^{N}|S_k|\left(e^{-\frac{k}{t_{n+1}}}-e^{-\frac{k}{t_n}}\right)\\
=\;&\left(e^{-\frac{1}{t_{n+1}}}-e^{-\frac{1}{t_n}}\right)\frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)}\sum_{k=0}^{N}|S_k|\sum_{\nu=0}^{k-1}e^{-\frac{\nu}{t_{n+1}}}e^{-\frac{k-\nu}{t_n}}\\
\leq\;&\left(e^{-\frac{1}{t_{n+1}}}-e^{-\frac{1}{t_n}}\right)\frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)}\sum_{k=0}^{N}|S_k|ke^{-\frac{k}{t_{n+1}}}.
\end{aligned}$$

Repeating the above argument we get

$$\left(e^{-\frac{1}{t_{n+1}}}-e^{-\frac{1}{t_n}}\right)\frac{(1-e^{-\frac{1}{t_n}})}{\varphi(t_n)}\leq\frac{1}{t_n^2\varphi(t_n)(\log n)^{\alpha+2}}$$

Using again the Chow-Teicher LIL for weighted means we get the bound

$$\limsup_{n\to\infty}\sum_{k=0}^{n}|S_k|k\leq c(\alpha)n^2(P_n^{(2)}\log\log P_n^{(2)})^{\frac{1}{2}}.$$

We now can repeat the previous argument splitting the sum this time at $N_2 = 2(\alpha+3)t_n\log\log t_n$.
■

We proceed using de Acosta's (k and t - dependent) truncation technique. Set

$$X_k = X_k^d(t)+X_k^u(t)$$

with

$$\begin{aligned}
X_k^d(t) &= X_k\mathbf{1}\left(|X_k|\leq\tau\frac{e^{\frac{k}{t}}}{p_k}\sqrt{\frac{2V(t)}{\log\log V(t)}}\right)\\
X_k^u(t) &= X_k\mathbf{1}\left(|X_k|>\tau\frac{e^{\frac{k}{t}}}{p_k}\sqrt{\frac{2V(t)}{\log\log V(t)}}\right)
\end{aligned}$$

where $\tau>0$ is a real number to be specified later. Let

$$p_X^u(t) := \sum_{k=0}^{N(t)}p_kX_k^u(t)e^{-\frac{k}{t}}.$$

LEMMA 3 *Let* (p_n) *satisfy conditions* (1.1) *and* (1.6) *with* $\alpha > -\frac{1}{2}$, *then we have:*

$$\lim_{n\to\infty} \frac{p_X^u(t_n)}{\varphi(t_n)} = 0 \quad a.s..$$

Proof: We start by showing

$$\lim_{n\to\infty} \varphi(t_n)^{-1} \sum_{k=0}^{t_n} p_k X_k^u(t_n) e^{-\frac{k}{t_n}} = 0 \quad a.s.. \tag{2.5}$$

We have

$$\left| \varphi(t_n)^{-1} \sum_{k=0}^{t_n} p_k X_k^u(t_n) e^{-\frac{k}{t_n}} \right| \leq \varphi(t_n)^{-1} \sum_{k=0}^{t_n} p_k |X_k| \mathbf{1} \left(|X_k| > \tau \frac{e^{\frac{k}{t}}}{p_k} \sqrt{\frac{2V(t_n)}{\log\log V(t_n)}} \right) e^{-\frac{k}{t_n}}.$$

Using Lemma A1 we see that

$$\sqrt{\frac{2V(t_n)}{\log\log V(t_n)}} \geq (\Gamma(2\alpha+1))^{\frac{1}{2}} \sqrt{\frac{t_n^{2\alpha+1} L(t_n)}{\log\log t_n}} =: h(t_n).$$

Hence

$$\mathbf{1} \left(|X_k| > \tau \frac{e^{\frac{k}{t_n}}}{p_k} \sqrt{\frac{2V(t_n)}{\log\log V(t_n)}} \right) \leq \mathbf{1} \left(|X_k| > \tau \frac{e^{\frac{k}{t_n}}}{p_k} h(t_n) \right).$$

In the range $k_0 \leq k \leq t_n$ we have $h(t_n) e^{k/t_n} \geq h(k)e$ and so (with k_0 large enough),

$$\sum_{k=k_0}^{t_n} p_k |X_k| \mathbf{1} \left(|X_k| > \tau \frac{e^{\frac{k}{t_n}}}{p_k} h(t_n) \right) e^{-\frac{k}{t_n}} \leq \sum_{k=k_0}^{t_n} p_k |X_k| \mathbf{1} \left(|X_k| > \tau \frac{e}{p_k} h(k) \right). \tag{2.6}$$

By Kronecker's Lemma, see [6] p.114, (2.5) will tend to 0 if

$$\sum_{k=k_0}^{\infty} |X_k| \mathbf{1} \left(|X_k| > \tau \frac{e}{p_k} h(k) \right) \frac{p_k}{\varphi(k)} < \infty \quad a.s..$$

This will follow if the expectation of the series is finite. Using the asymptotic results for regularly varying functions we get

$$\begin{aligned}
& \sum_{k=k_0}^{\infty} \frac{p_k}{\varphi(k)} E\left(|X_k| \mathbf{1}\left(|X_k| > \tau \frac{e}{p_k} h(k) \right)\right) \\
= & \sum_{k=k_0}^{\infty} \frac{p_k}{\varphi(k)} \sum_{j=k}^{\infty} E\left(|X_k| \mathbf{1}\left(e\tau \frac{h(j)}{p_j} < |X_k| \le e\tau \frac{h(j+1)}{p_{j+1}} \right)\right) \\
\le & \sum_{k=k_0}^{\infty} \frac{p_k}{\varphi(k)} \sum_{j=k}^{\infty} \frac{e\tau h(j+1)}{p_{j+1}} P\left(e\tau \frac{h(j)}{p_j} < |X| \le e\tau \frac{h(j+1)}{p_{j+1}} \right) \\
\le & \sum_{j=k_0}^{\infty} \frac{e\tau h(j+1)}{p_{j+1}} P\left(e\tau \frac{h(j)}{p_j} < |X| \le e\tau \frac{h(j+1)}{p_{j+1}} \right) \sum_{k=k_0}^{j} \frac{p_k}{\varphi(k)} \\
\le & \; e\tau \sum_{j=k_0}^{\infty} \frac{jL(j)}{\log\log j} P\left(e\tau \sqrt{\frac{jL(j)}{\log\log j}} < |X| \le e\tau \sqrt{\frac{(j+1)L(j+1)}{\log\log(j+1)}} \right) < \infty,
\end{aligned}$$

where the convergence of the series follows from the fact that $E(X^2) < \infty$ is equivalent to the convergence of series of type $\sum kP(\sqrt{k} \le |X| \le \sqrt{k+1})$.

We are left with showing

$$\frac{1}{\varphi(t_n)} \sum_{k=t_n+1}^{t_n \log t_n} p_k e^{-\frac{k}{t_n}} X_k^u(t_n) \to 0 \quad a.s.. \tag{2.7}$$

Set $S_j^u(t_n) = \sum_{k=0}^{j} p_k X_k^u(t_n)$. We use the notation $N(t_n) := t_n \log t_n$ and omit the argument t_n in the following. By Abel's partial summation we get

$$\sum_{k=t_n+1}^{N} p_k e^{-\frac{k}{t_n}} X_k^u = S_N^u \exp\{-\log t_n\} - S_{t_n+1}^u \exp\left\{ -\frac{t_n+1}{t_n} \right\} + \sum_{k=t_n+1}^{N} (1-e^{-\frac{1}{t_n}}) e^{-\frac{k}{t_n}} S_k^u.$$

Analysing the above argument we see that we have in fact shown that for any sequence (k_ν) with $k_\nu \to \infty$ $(\nu \to \infty)$ we have

$$\frac{1}{\varphi(k_\nu)} S_{k_\nu}^u(t_n) \to 0 \quad (\nu \to \infty) \; a.s.. \tag{2.8}$$

Using Lemma A1 and the above result we get

$$\frac{S_N^u e^{-\log t_n}}{\varphi(t_n)} \le \left(\frac{S_N^u}{\varphi(N)} \right) \left(\frac{\varphi(N)}{\varphi(t_n) t_n} \right) \to 0 \quad a.s..$$

So we are left with discussing the sum, i.e.

$$Z_n := (t_n\varphi(t_n))^{-1}\sum_{k=t_n}^{N} e^{-\frac{k}{t_n}}|S_k^u|.$$

With $Y_k := |S_k^u|/\varphi(k)$ we have

$$Z_n = (t_n\varphi(t_n))^{-1}\sum_{k=t_n}^{N} e^{-\frac{k}{t_n}}\varphi(k)Y_k \le \left(\max_{t_n\le k\le t_n\log t_n} Y_k\right)(t_n\varphi(t_n))^{-1}\sum_{k=t_n}^{N} e^{-\frac{k}{t_n}}\varphi(k).$$

Since φ is regularly varying with index $\alpha+\frac{1}{2}$, we have by Lemma A2

$$\sum_{k=t_n}^{N} e^{-\frac{k}{t_n}}\varphi(k) \le c(\alpha)t_n^{\alpha+\frac{3}{2}}(\log\log t_n)^{\frac{1}{2}} = \boldsymbol{O}\left(\varphi(t_n)t_n\right).$$

Now by (2.8) $\sup_{k\ge t_n} Y_k \to 0 \quad a.s$ and hence we obtain

$$Z_n \to 0 \ \ (n\to\infty) \quad a.s..$$

■

LEMMA 4 *We have*

$$\limsup_{n\to\infty}\frac{p_X^d(t_n)}{\varphi(t_n)} \le 1 \ a.s. \quad \textit{and} \quad \liminf_{n\to\infty}\frac{p_X^d(t_n)}{\varphi(t_n)} \ge -1 \ a.s..$$

Proof: Notice first that

$$E\left(\frac{p_X^d(t_n)}{\varphi(t_n)}\right) = -E\left(\frac{p_X^u(t_n)}{\varphi(t_n)}\right) \to 0 \quad (n\to\infty).$$

Let $Y_k^d(t) := X_k^d(t) - E(X_k^d(t))$, then $Y_k^d(t)$ has mean $=0$ and variance

$$E((Y_k^d(t))^2) = E((X_k^d(t) - E(X_k^d(t)))^2) \le E(X_k^2) = 1.$$

Furthermore we have

$$p_Y^d(t) = \sum_{k=0}^{N(t_n)} Y_k^d(t)p_k e^{-\frac{k}{t_n}} = p_X^d(t) - E(p_X^d(t)).$$

We compute an estimate of the moment generating function by using the standard exponential inequalities, see [2] p.276 and p.278,

$$\begin{aligned} E\left(\exp\left\{\frac{sp_Y^d(t_n)}{\varphi(t_n)}\right\}\right) &= \prod_{k=0}^{N(t_n)} E\left(\exp\left\{\frac{sp_kY_k(t_n)e^{-\frac{k}{t_n}}}{\varphi(t_n)}\right\}\right) \\ &\leq \prod_{k=0}^{N(t_n)}\left(1+\frac{s^2p_k^2e^{-\frac{2k}{t_n}}}{2\varphi(t_n)^2}E\left(\exp\left\{\frac{|s||Y_k(t)|p_ke^{-\frac{k}{t_n}}}{\varphi(t_n)}\right\}\right)\right). \end{aligned}$$

Now $|Y_k(t_n)| \leq \tau\sqrt{2V(t_n)/\log\log V(t_n)}e^{\frac{k}{t_n}}/p_k$, and using the inequality $1+a \leq e^a$ we get

$$\begin{aligned} E\left(\exp\left\{\frac{sp_Y^d(t_n)}{\varphi(t_n)}\right\}\right) &\leq \prod_{k=0}^{N(t_n)}\left(1+\frac{s^2p_k^2e^{-\frac{2k}{t_n}}}{2\varphi(t_n)^2}\exp\left\{\frac{|s|\tau}{\log\log V(t_n)}\right\}\right) \\ &\leq \exp\left\{\frac{s^2}{2}\varphi(t_n)^{-2}\sum_{k=0}^{t_n\log t_n} p_k^2e^{-\frac{2k}{t_n}}\exp\left\{\frac{|s|\tau}{\log\log V(t_n)}\right\}\right\} \\ &\leq \exp\left\{\frac{s^2}{4\log\log V(t_n)}\exp\left\{\frac{|s|\tau}{\log\log V(t_n)}\right\}\right\}. \end{aligned}$$

Now we use the exponential Markov-inequality to get for all $s > 0, \gamma > 0$

$$\begin{aligned} P\left(\frac{p_Y(t_n)}{\varphi(t_n)} \geq 1+\gamma\right) &= P\left(\exp\left\{\frac{sp_Y(t_n)}{\varphi(t_n)}\right\} \geq e^{s(1+\gamma)}\right) \\ &\leq e^{-s(1+\gamma)}E\left(\exp\left\{\frac{sp_Y(t_n)}{\varphi(t_n)}\right\}\right). \end{aligned}$$

Hence for $s = 2(1+\gamma)\log\log V(t_n)$ we obtain

$$P\left(\frac{p_Y(t_n)}{\varphi(t_n)} \geq 1+\gamma\right) \leq \exp\left\{-(1+\gamma)^2\log\log V(t_n)\left(2-e^{2\tau(1+\gamma)}\right)\right\}.$$

For any $\gamma > 0$ we can choose $\tau > 0$ small enough such that

$$(1+\gamma)^2\left(2-e^{2\tau(1+\gamma)}\right) = 1+\delta > 1.$$

Now we can bound the above probability by the general term of a summable series and using the Borel-Cantelli Lemma we obtain that the $\limsup \leq 1$ $a.s.$. The argument for the $\liminf$ is analogous.

■

We close this section by putting together all our Lemmas to the final result.

PROPOSITION 1 *Let* $X_0, X_1, \ldots$ *be an i.i.d. sequence of random variables with* $E(X_0) = 0, E(X_0^2) = 1$ *and let* (p_k) *be a sequence of weights satisfying* (1.1) *and* (1.6) *with* $\alpha > -\frac{1}{2}$ *. Then*

$$\limsup_{t\to\infty} \frac{\sum_{k=0}^{\infty} p_k X_k e^{-\frac{k}{t}}}{\varphi(t)} \leq 1 \quad a.s.$$

and

$$\liminf_{t\to\infty} \frac{\sum_{k=0}^{\infty} p_k X_k e^{-\frac{k}{t}}}{\varphi(t)} \geq -1 \quad a.s..$$

3 *A Converse to the Law of Iterated logarithm*

PROPOSITION 2 *Let* (X_n) *be a sequence of i.i.d. random variables and* (p_n) *be a sequence of weights satisfying* (1.1) *and* (1.6) *with* $\alpha > -\frac{1}{2}$ *. Then*

$$P\left(\limsup_{t\to\infty} \frac{|p_X(t)|}{\varphi(t)} < \infty\right) > 0$$

implies $E(X_0) = 0$ *and* $E(X_0^2) < \infty$ *.*

Proof: We adapt an argument by Feller ([2, 19], pp.297–301, pp.297–298).
By the $0-1$-Law we get

$$\limsup_{t\to\infty} \frac{|p_X(t)|}{\varphi(t)} < \infty \quad a.s..$$

By Lemma A1

$$\varphi(t) = \sqrt{2V(t)\log\log V(t)} = \boldsymbol{O}\left(t^{\alpha+\frac{1}{2}} L(t)(\log\log t)^{\frac{1}{2}}\right) = \boldsymbol{o}\left(t^{\alpha+1}L(t)\right) = \boldsymbol{O}\left(p(t)\right),$$

hence using the converse of the Strong Law of Large Numbers for power series methods, c.f. [13, 14], it follows that $E(|X_0|) < \infty$ and $E(X_0) = 0$. By the usual symmetrisation argument, we can w.l.o.g. assume that the X_n are symmetric.

We suppose that $E(X_0^2) = \infty$ and show that this will lead to a contradiction.

Fix $M > 0$ and define $Y_n := X_n \mathbf{1}(|X_n| \le \alpha_M)$, where we choose α_M in such a way that $E(Y_n^2) > M$.

By symmetry the $Y_0, Y_1, \dots$ are i.i.d. with $E(Y_n) = 0$ and $E(Y_n^2) < \infty$. Hence they obey the LIL of Proposition 1 and using in particular the same sequence (t_n) as above we have

$$P\left(p_Y(t_n) > \sqrt{2MV(t_n)\log\log V(t_n)}\ i.o.\right) = 1.$$

Let $Z_n := X_n \mathbf{1}(|X_n| > \alpha_M)$ and $\tilde{X}_n := Y_n - Z_n$. By symmetry X_n and $\tilde{X}_n$ have the same distribution for each n. Therefore the two events

$$A_+^M := \left\{p_Y(t_n) > \sqrt{2MV(t_n)\log\log V(t_n)}, p_X(t_n) - p_Y(t_n) \ge 0\ i.o.\right\}$$

$$A_-^M := \left\{p_Y(t_n) > \sqrt{2MV(t_n)\log\log V(t_n)}, p_X(t_n) - p_Y(t_n) \le 0\ i.o.\right\}$$

have the same probability, since they are given by $\{X \in B\}$ resp. $\{\tilde{X} \in B\}$ with B a Borel-set in $\mathbb{R}^{\mathbb{N}}$, compare the argument on p.299 in [2]. Now

$$A_+^M \cup A_-^M = \left\{p_Y(t_n) > \sqrt{2MV(t_n)\log\log V(t_n)},\ i.o.\right\}.$$

Using the subadditivity of probability measures this implies

$$P(A_+^M) + P(A_-^M) \ge P(A_+^M \cup A_-^M) = 1,$$

and hence $P(A_+^M) \ge \frac{1}{2}$. So we obtain for arbitrary $M > 0$ that $P(A_+^M) \ge \frac{1}{2}$. Using the inclusion

$$\begin{aligned} &\left\{\limsup_{n\to\infty} \frac{p_X(t_n)}{\sqrt{2V(t_n)\log\log V(t_n)}} = +\infty\right\} \\ &\supseteq \bigcap_{M=1}^{\infty} \left\{p_Y(t_n) > \sqrt{2MV(t_n)\log\log V(t_n)}, p_X(t_n) - p_Y(t_n) \ge 0\ i.o.\right\} \end{aligned}$$

and that the sequence A_+^M is decreasing and the continuity of the probability measure we get

$$P\left(\limsup_{n\to\infty}\frac{p_X(t_n)}{\sqrt{2V(t_n)\log\log V(t_n)}}=+\infty\right)$$
$$\geq P\left(\bigcap_{M=1}^{\infty}A_+^M\right)=\lim_{M\to\infty}P(A_+^M)\geq\frac{1}{2}.$$

Therefore the $0-1$-Law implies that

$$P\left(\limsup_{n\to\infty}\frac{p_X(t_n)}{\sqrt{2MV(t_n)^2\log\log V(t_n)^2}}=+\infty\right)=1,$$

which is a contradiction.
■

4 *The cluster set*

PROPOSITION 3 *To each $\rho\in[-1,1]$ there exists a sequence $t_n\to\infty$ such that*

$$\lim_{n\to\infty}\frac{p_X(t_n)}{\varphi(t_n)}=\rho\quad a.s..$$

Proof: Define $(\log n)! := \prod_{k=2}^{n}\log k$ and with this notation

$$\begin{aligned}
t_n &:= n!(\log n)!(\log\log n)!,\\
N_2 &:= N(t_n) := t_n\log t_n \sim n!(\log n)!(\log\log n)!n\log n,\\
N_1 &:= N_1(t_n) := t_n(\log n)^{-1} = n!(\log n)!(\log\log n)!(\log n)^{-1}.
\end{aligned}$$

Observe that $N_2/N_1\sim n(\log n)^2$. We split $p_X(t_n)$ in three parts, namely

$$\begin{aligned}
p_X(t_n) &= \sum_{k=0}^{N_1-1}p_kX_ke^{-\frac{k}{t_n}}+\sum_{k=N_1}^{N_2}p_kX_ke^{-\frac{k}{t_n}}+\sum_{k=N_2+1}^{\infty}p_kX_ke^{-\frac{k}{t_n}}\\
&=: p_X^{(1)}(t_n)+p_X^{(2)}(t_n)+p_X^{(3)}(t_n).
\end{aligned}$$

By Lemma 1 we immediatly get

$$\lim_{n\to\infty} \frac{p_X^{(3)}(t_n)}{\varphi(t_n)} = 0 \quad a.s..$$

Furthermore

$$\begin{aligned}
\frac{p_X^{(1)}(t_n)}{\varphi(t_n)} &= \frac{\left|\sum_{k=0}^{N_1-1} p_k X_k e^{-\frac{k}{t_n}}\right|}{\sqrt{2V(t_n)\log\log V(t_n)}} \\
&\leq \frac{\left|\sum_{k=0}^{N_1-1} p_k X_k e^{-\frac{k}{t_n}}\right|}{\sqrt{2\sum_{k=0}^{N_1-1} p_k^2 e^{-\frac{2k}{t_n}} \log\log V(t_n)}} \times \sqrt{\frac{\sum_{k=0}^{N_1-1} p_k^2 e^{-\frac{2k}{t_n}}}{V(t_n)}}.
\end{aligned}$$

By the Chow-Teicher LIL [5], we have

$$\frac{\left|\sum_{k=0}^{N_1-1} p_k X_k\right|}{\sqrt{2\sum_{k=0}^{N_1-1} p_k^2 \log\log N_1}} \leq 1 \ a.s..$$

Since in our range of consideration $e^{-\frac{k}{t_n}}$ is always close to 1, we can repeat their arguments to get

$$\frac{\left|\sum_{k=0}^{N_1} p_k X_k e^{-\frac{k}{t_n}}\right|}{\sqrt{2\sum_{k=0}^{N_1} p_k^2 e^{-\frac{2k}{t_n}} \log\log V(t_n)}} \leq 1 \ a.s..$$

On the other hand by Lemma A1 we get

$$\frac{\sum_{k=0}^{N_1-1} p_k^2 e^{-\frac{2k}{t_n}}}{V(t_n)} \leq c(\alpha) \frac{(t_n/\log t_n)^{2\alpha+1} L(t_n/\log t_n)}{t_n^{2\alpha+1} L(t_n)} \to 0 \ \ (n\to\infty).$$

Hence

$$\lim_{n\to\infty} \frac{p_X^{(1)}(t_n)}{\varphi(t_n)} = 0 \quad a.s..$$

Since for large n $N_1(t_{n+1}) > N_2(t_n)$ we see that $p_X^{(2)}(t_{n+1})$ and $p_X^{(2)}(t_n)$ are independent for such n. If we set

$$A_n^\epsilon := \left\{ \left| \frac{p_X^{(2)}(t_n)}{\varphi(t_n)} - \rho \right| < \epsilon \right\}$$

for $\epsilon > 0$ arbitrary, we are left with showing $P(A_n^\epsilon \ i.o.) = 1$.

So by the Borel-Cantelli Lemma our proof is finished, if we can show

Lemma 5

$$\sum_{n=2}^{\infty} P(A_n^\epsilon) = \infty \textit{ for each } \epsilon > 0.$$

Proof: We represent $p_X^{(2)}(t_n)$ as the average of $q_n := \tau^{-2} 2 \log\log V(t_n)$ independent random variables, each of which converges in distribution to a Gaussian random variable of variance 1 by the Embrechts-Maejima Central Limit Theorem, see [7]. To do this we set

$$\sigma_n^2 := \sum_{k=N_1}^{N_2} p_k^2 e^{-\frac{2k}{t_n}}$$

and define numbers p_{ln} for $0 \le l \le q_n$ and each n recursivly by

$$\begin{aligned}
p_{0n} &:= N_1, \\
p_{ln} &:= \inf\left\{ m : \sum_{k=p_{l-1,n}+1}^{m} p_k^2 e^{-\frac{2k}{t_n}} \ge \frac{\sigma_n^2}{q_n} \right\}, \\
p_{q_n n} &:= N_2.
\end{aligned}$$

Since

$$\frac{p_k^2 e^{-\frac{2k}{t_n}}}{\sigma_n^2} \le c_1(\alpha) \frac{(t_n \log t_n)^{2\alpha} L^2(t_n \log t_n)}{t_n^{2\alpha+1} L(t_n)} \le c_2(\alpha) \frac{(\log t_n)^{2\alpha} L(t_n \log t_n)}{t_n} \to 0 \ \ (n \to 0)$$

uniformly in k the $\{p_{ln}; n \in I\!N,\ 0 \le l \le q_n\}$ are well-defined. Furthermore

$$p_{ln} - p_{l-1,n} \ge c(\alpha) \frac{t_n}{(\log t_n)^{2\alpha} L(t_n \log t_n) \log\log t_n} \to \infty \ \ (n \to \infty)$$

uniformly in l. We define

$$\tilde{Z}_{ln} := \sum_{k=p_{l-1,n}+1}^{p_{ln}} p_k X_k e^{-\frac{k}{t_n}},$$

then $E(\tilde{Z}_{ln}) = 0$ and

$$E(\tilde{Z}_{ln}^2) = \sum_{k=p_{l-1,n}+1}^{p_{ln}} p_k^2 e^{-\frac{2k}{t_n}} \sim \frac{\sigma_n^2}{q_n}.$$

Hence

$$Z_{ln} := \sqrt{\frac{q_n}{\sigma_n^2}} \sum_{k=p_{l-1,n}+1}^{p_{ln}} p_k X_k e^{-\frac{k}{t_n}}$$

has mean 0 and variance ≈ 1. Furthermore

$$\sum_{l=1}^{q_n} Z_{ln} = \frac{\sqrt{q_n}}{\sigma_n} \sum_{k=N_1}^{N_2} p_k X_k e^{-\frac{k}{t_n}} = \frac{\sqrt{q_n}}{\sigma_n} p_X^{(2)}(t_n).$$

Therefore

$$\frac{p_X^{(2)}(t_n)}{\varphi(t_n)} = \frac{\sigma_n \sum_{l=1}^{q_n} Z_{ln}}{\sqrt{q_n 2V(t_n)\log\log V(t_n)}} = \frac{\sum_{l=1}^{q_n} Z_{ln}}{\tau q_n}(1 + \boldsymbol{o}(1)).$$

Hence for any interval $[c,d] \subset [-1,1]$ with $\rho \in [c,d]$ we can use the independence of the Z_{ln} to get

$$P\left(\frac{p_X^{(2)}(t_n)}{\varphi(t_n)} \in [c,d]\right) \geq \prod_{l=1}^{q_n} P\left(Z_{ln} \in \tau[c,d]\right).$$

For the Z_{ln} the Embrechts-Maejima CLT [7], Theorem 1 (see also the Remark), applies and we have uniformly in l

$$P\left(Z_{ln} \in \tau[c,d]\right) \to \gamma(\tau[c,d]) \ \ (n \to \infty)$$

with

$$\gamma(\tau[c,d]) = \frac{1}{\sqrt{2\pi}} \int_{c\tau}^{d\tau} \exp\{-x^2/2\} dx.$$

Now let $c = \rho - \epsilon, d = \rho + \epsilon$, then by ([2], pp.294,295)

$$\gamma(\tau[c,d]) \geq \exp\{-(\tau^2\rho^2/2) + \log(\gamma([-\tau\epsilon, \tau\epsilon]))\}.$$

Therefore

$$P\left(\frac{p_X^{(2)}(t_n)}{\varphi(t_n)}\in[c,d]\right) \geq \exp\{-q_n(\tau^2\rho^2/2)+q_n\log(\gamma([-\tau\epsilon,\tau\epsilon]))\}$$

$$= \exp\{\log\log V(t_n)^2(-\rho^2+\tau^{-2}2\log(\gamma([-\tau\epsilon,\tau\epsilon])))\},$$

and we can choose τ large enough, so that

$$\rho^2-\tau^{-2}2\log(\gamma([-\tau\epsilon,\tau\epsilon]))=\delta<1.$$

Since by Lemma A1 $\log\log V(t_n)\sim\log(n\log n)$, we get

$$P\left(\frac{p_X^{(2)}(t_n)}{\varphi(t_n)}\in[c,d]\right)\sim\frac{1}{(n\log n)^\delta}$$

and this is the general term of a divergent series.
■

REMARK: Applicability of the Embrechts-Maejima CLT.
We set

$$W_{ln}:=\frac{\sum\limits_{k=p_{l-1,n}+1}^{p_{ln}} p_kX_ke^{-\frac{k}{t_n}}}{\sum\limits_{k=p_{l-1,n}+1}^{p_{ln}} p_ke^{-\frac{k}{t_n}}}=\sum_{k=0}^{\infty}c_k(ln)X_k.$$

Then we have $\sum\limits_{k=0}^{\infty}c_k(ln)=1$ and

$$d_{ln}^2:=\sum_{k=0}^{\infty}c_k(ln)=\frac{\sum\limits_{k=p_{l-1,n}+1}^{p_{ln}} p_k^2e^{-\frac{2k}{t_n}}}{\left(\sum\limits_{k=p_{l-1,n}+1}^{p_{ln}} p_ke^{-\frac{k}{t_n}}\right)^2}\leq c(\alpha)\frac{t_n^{2\alpha+1}L^2(t_n)}{t_n^{2\alpha+2}L^2(t_n)\log\log t_n}\to 0\ \ (n\to\infty).$$

Since furthermore

$$\frac{\sup_k c_k(ln)}{d_{ln}}=\frac{1}{d_{ln}}\sup_k\left\{\frac{p_ke^{-\frac{k}{t_n}}}{\sum\limits_{j=p_{l-1,n}+1}^{p_{ln}} p_je^{-\frac{j}{t_n}}}\right\}\to 0\ \ (n\to\infty)$$

the Embrechts-Maejima [7] CLT applies to W_{ln}, which means that $Z_{ln} = \frac{1}{d_{ln}} W_{ln}$ converges in law to a Gaussian random variable with mean 0 and variance 1.
■

5 *The LIL for* $(N_{\beta p})$ *-methods*

5.1 *The upper bound*

We use the following notations:

$$\begin{aligned} S_n^\beta &:= \sum_{j=0}^{n} \binom{n-j+\beta-1}{n-j} p_j X_j, \\ U_n^\beta &:= \sum_{j=0}^{n} \left(\binom{n-j+\beta-1}{n-j} \right)^2 p_j^2, \\ a_n &:= \sqrt{2U_n^\beta \log\log U_n^\beta}. \end{aligned}$$

Observe that by Lemma A1 $a_n^2 \sim 2B(2\alpha+1, 2\beta-1) n^{2(\alpha+\beta)-1} \log\log n$. We want to show that

$$\limsup_{n\to\infty} \frac{S_n^\beta}{a_n} \le 1 \quad a.s..$$

To do this, we use a general theorem on the Law of Iterated logarithm for triangular matrix methods by Stadtmüller [18] (we use notation of this paper in the following). Since U_n^β is regularly varying (with index $2(\alpha+\beta)-1$), we only have to show that the conditions (1.4) and (1.5a) of Stadtmüller's Theorem 1 hold true. In our case (1.4) reduces to

$$(5.1) \qquad \lim_{\rho\to 1+} \liminf_{n\to\infty} \min_{m:1\le n/m\le\rho} \sum_{j=0}^{m} \binom{m-j+\beta-1}{m-j} \binom{n-j+\beta-1}{n-j} p_j^2 / U_m^\beta = 1.$$

Choosing $m = n$ we immediately see that in (5.1) the inequality ≤ 1 holds true. Using the properties of the binomial coefficients, we get that the left-hand side of (5.1) is greater or equal to

$$\lim_{\rho\to 1+} \liminf_{n\to\infty} \left(1 + \frac{(\beta-1)}{n} \right)^{(\rho-1)n} = \lim_{\rho\to 1+} \exp\{(\beta-1)(\rho-1)\} = 1.$$

Similarly, condition (1.5a) reduces in our context to

$$\frac{1}{n^{\alpha+\beta-1}L(n)}\sum_{j=2}^{n}\left|\binom{n-j+\beta-1}{n-j}p_j-\binom{n-j+\beta}{n-j+1}p_{j-1}\right| = \boldsymbol{O}(1)$$

and this can be seen by using the well-known identities for the binomial coefficients.

5.2 *The cluster set*

We want to show:
To each $\rho\in[-1,1]$ there exists a sequence $n_k\to\infty$ such that

$$\lim_{n\to\infty}\frac{S^{\beta}_{n_{k+1}}}{a_{n_{k+1}}}=\rho \quad a.s..$$

To prove this we set

$$n_k := k^k.$$

Using the triangle inequality we get

$$\begin{aligned}\left|\tfrac{1}{a_{n_{k+1}}}S^{\beta}_{n_{k+1}}\right| &\le \left|\frac{1}{a_{n_{k+1}}}\sum_{j=0}^{n_k}\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j}p_jX_j\right| \\ &+ \left|\frac{1}{a_{n_{k+1}}}\sum_{j=n_k+1}^{n_{k+1}}\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j}p_jX_j\right| \\ &:= S_k^{(1)}+S_k^{(2)}.\end{aligned}$$

To estimate $S_k^{(1)}$ we can adjust Stadtmüller's [18] LIL to get

$$\limsup_{k\to\infty}\frac{\sum_{j=0}^{n_k}\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j}p_jX_j}{\sqrt{2s_k^2\log\log s_k^2}}\le 1 \;\; a.s.$$

and using the same techniques as in Lemmas A1 and A2 we have

$$\begin{aligned} s_k^2 &:= \sum_{j=0}^{n_k}\left(\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j}\right)^2 p_j^2 \\ &\le c n_{k+1}^{2\alpha+2\beta-1}L(n_{k+1})\int_0^{\frac{n_k}{n_{k+1}}}(1-t)^{(2\alpha+1)-1}t^{(2\beta-1)-1}dt.\end{aligned}$$

Since

$$\frac{2s_k^2 \log\log s_k}{a_{n_{k+1}}^2} \le \int_0^{\frac{n_k}{n_{k+1}}} (1-t)^{(2\alpha-1)-1} t^{(2\beta+1)-1} dt \to 0 \quad (k\to\infty)$$

we get

$$S_{n_{k+1}}^{(1)} \to 0 \; a.s..$$

Now for $\epsilon > 0$ let

$$A_k^\epsilon := \left\{ \left| \frac{1}{a_{n_{k+1}}} \sum_{j=n_k+1}^{n_{k+1}} \binom{n_{k+1}-j+\beta-1}{n_{k+1}-j} p_j X_j - \rho \right| < \epsilon \right\}.$$

Then (A_k^ϵ) is a sequence of independent events and by the Borel-Cantelli Lemma we are done, if we can show that

$$\sum_{k=0}^{\infty} P(A_k^\epsilon) = \infty.$$

We use the same ideas as in §4 and represent $S_{n_{k+1}}^{(2)}$ as an average of $q_k := \tau^{-2} 2\log\log U_{n_k}^\beta$ random variables, each of which converges to a Gaussian random variable with mean 0 and variance 1. Set

$$\sigma_k^2 := \sum_{j=n_k+1}^{n_{k+1}} \left(\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j} \right)^2 p_j^2$$

and choose numbers p_{lk} for each k recursively for $0 \le l \le q_k$ by

$$\begin{aligned} p_{0k} &:= n_k; \\ p_{lk} &:= \inf\left\{ m : \sum_{j=p_{l-1,k}+1}^{m} \left(\binom{n_{k+1}-j+\beta-1}{n_{k+1}-j} \right)^2 p_j^2 \ge \frac{\sigma_k^2}{q_k} \right\}; \\ p_{q_k,k} &:= n_{k+1}-1. \end{aligned}$$

Using the asymptotic results in much the same way as in §4 in the proof of Lemma 5 we see that the p_{lk} are well-defined and $p_{l-1,k} - p_{lk} \to \infty$ $(k\to\infty)$ uniformly in l.

We set

$$Z_{lk} := \frac{\sqrt{q_k}}{\sigma_k^2} \sum_{j=p_{l-1,k}+1}^{p_{lk}} \binom{n_{k+1}-j+\beta-1}{n_{k+1}-j} p_j X_j$$

and have $E(Z_{lk}) = 0$ and $E(Z_{lk}^2) \sim 1$. Furthermore

$$\frac{S_k^{(2)}}{a_{n_{k+1}}} = \frac{\sigma_k}{\sqrt{q_k}} \frac{\sum_{l=1}^{q_k} Z_{lk}}{\sqrt{2U_{n_k}^\beta \log\log(U_{n_k}^\beta)}} = \frac{\sum_{l=1}^{q_k} Z_{lk}}{\tau q_k}(1 + \boldsymbol{o}(1)).$$

Hence for any interval $[c, d] \subset (-1, 1)$ with $\rho \in [c, d]$ we have

$$P\left(\frac{S_k^{(2)}}{a_{n_{k+1}}} \in [c, d]\right) = \prod_{l=1}^{q_k} P(Z_{lk} \in \tau[c, d]).$$

Now choosing $c = \rho - \epsilon, d = \rho + \epsilon$ and using the Embrechts-Maejima CLT [7] for each Z_{lk} we get

$$\begin{aligned} P\left(\frac{S_k^{(2)}}{a_{n_{k+1}}} \in [c, d]\right) &\geq \exp\left\{q_k(-\tau^2\rho^2/2) + q_k \log\gamma([-\tau\epsilon, \tau\epsilon])\right\} \\ &= \exp\left\{\log\log U_{n_k}^\beta(-\rho^2 + \log\gamma([-\tau\epsilon, \tau\epsilon])\right\} \end{aligned}$$

and since $\log\log U_{n_k}^2 \sim \log(k \log k)$ we see that this is the general term of a divergent series, if we choose τ large enough.

5.3 *The converse of the LIL*

Thsi follows by exactly the same arguments as in the power series case.

6 *Appendix*

Throughout this paragragh we assume that the sequence (p_n) satisfies condition (1.6), i.e.

$$p_n = n^\alpha L(n) \quad (n \to \infty)$$

with $\alpha > -1$ and $L(.)$ a slowly varying function, see [17, 3] for details on such functions.

LEMMA A1 *We have the following asymptotic properties* *(i)* $p(t) \sim \Gamma(\alpha+2)P(t) \sim \Gamma(\alpha+2)t^{\alpha+1}L(t) \quad (t \to \infty)$;

(ii) $V(t) \sim \Gamma(2\alpha+2)P^{(2)}(t) \sim \Gamma(2\alpha+2)t^{2\alpha+1}L^2(t) \quad (t \to \infty)$;

(iii) $\sum_{k=0}^{n} \binom{n-k+\beta-1}{n-k} p_k \sim B(\alpha+1,\beta)n^{\alpha+\beta}L(n) \quad (n \to \infty)$;

(iv) $\sum_{k=0}^{n} \left(\binom{n-k+\beta-1}{n-k}\right)^2 p_k^2 \sim B(2\alpha+1,2\beta-1)n^{2(\alpha+\beta)-1}L^2(n) \quad (n \to \infty)$.

Proof:
(i),(ii) are basically contained in [11], see also [12] for a detailed proof.
(iii),(iv) are proved in [15], Lemma 1 based on classical results from [9, 20], Theorem 42 resp. Theorem 2.3.1.

LEMMA A2 *Let* $h : [0,\infty) \to I\!N$ *with* $t = \boldsymbol{O}(h(t)) \quad (t \to \infty)$. *Then for* $t \geq t_0 (= t_0(L(.)))$

$$\sum_{k=h(t)}^{\infty} p_k e^{-\frac{k}{t}} \leq [\alpha+2][\alpha+2]!P(h(t))\exp\{-h(t)/t\}.$$

Proof:

$$\begin{aligned}\sum_{k=h(t)}^{\infty} p_k e^{-\frac{k}{t}} &= p_{h(t)}e^{-\frac{h(t)}{t}} + \int_{h(t)}^{\infty} e^{-\frac{x}{t}}dP(x)\\ &= p_{h(t)}e^{-\frac{h(t)}{t}} + P(x)e^{-\frac{x}{t}}\Big|_{h(t)}^{\infty} + \frac{1}{t}\int_{h(t)}^{\infty} e^{-\frac{x}{t}}P(x)dx.\end{aligned}$$

Using the substitution $u := x/t$ and the fact that $t^{-\gamma}L(t) \to 0 \ (t \to \infty)$ asymptotically non increasing for each $\gamma > 0$ [3], Theorem 1.5.4, Proposition 1.3.6

(v), we obtain

$$\begin{aligned}
\frac{1}{t}\int_{h(t)}^{\infty} e^{-\frac{x}{t}} P(x)dx &= \int_{h(t)/t}^{\infty} e^{-u} P(ut)du \\
&= t^{[\alpha+2]} \int_{h(t)/t}^{\infty} e^{-u} u^{[\alpha+2]} \Big((ut)^{\alpha+1-[\alpha+2]} L(ut)\Big)\, du \\
&\le t^{[\alpha+2]} \sup_{u \ge h(t)/t} \Big\{(ut)^{\alpha+1-[\alpha+2]} L(ut)\Big\} \int_{h(t)/t}^{\infty} e^{-u} u^{[\alpha+2]} du \\
&\le t^{[\alpha+2]} (h(t))^{\alpha+1-[\alpha+2]} L(h(t)) \sum_{k=0}^{[\alpha+2]} \frac{[\alpha+2]!}{k!} \left(\frac{h(t)}{t}\right)^{\alpha} e^{-\frac{h(t)}{t}} \\
&\le [\alpha+2][\alpha+2]!(h(t))^{\alpha+1} L(h(t)) e^{-\frac{h(t)}{t}}.
\end{aligned}$$

Furthermore we have for the first part

$$\left| p_{h(t)} e^{-\frac{h(t)}{t}} - P_{h(t)} e^{-\frac{h(t)}{t}} \right| \sim P_{h(t)} e^{-\frac{h(t)}{t}} \ (t \to \infty).$$

Therefore we obtain the proposed bound for t large enough.
■

LEMMA A3 *For* $t_n := \exp\{-n/(\log n)^\gamma\}$, $\gamma > 1$, *we have*

$$\exp\left\{\frac{-1}{(\log(n+1))^\gamma}\right\} \le \frac{t_n}{t_{n+1}} \le \exp\left\{\frac{-1}{2(\log n)^\gamma}\right\}.$$

Proof: Set $f(x) := x/(\log x)^\gamma$ then $f'(x) = \frac{1}{(\log x)^\gamma}\left(1 - \frac{\gamma}{\log x}\right) \ge 0$ for $x \ge e^\gamma$ and $f''(x) = \frac{\gamma}{x(\log x)^{\gamma+1}}\left(1 - \frac{\gamma}{\log x}\right) \le 0$ for $x \ge e^{\gamma+1}$. Therefore we get by the mean-value-theorem using the monotonicity of f' that $f'(n) \ge f(n+1) - f(n) \ge f'(n+1)$ and so

$$\begin{aligned}
\frac{t_n}{t_{n+1}} &= \exp\{f(n+1) - f(n)\} \le \exp\{f'(n)\} \\
&= \exp\left\{\frac{1}{(\log n)^\gamma}\left(1 - \frac{\gamma}{\log n}\right)\right\} \le \exp\left\{\frac{1}{2(\log n)^\gamma}\right\}.
\end{aligned}$$

$$
\begin{aligned}
\frac{t_n}{t_{n+1}} &= \exp\{f(n+1) - f(n)\} \geq \exp\{f'(n+1)\} \\
&= \exp\left\{\frac{1}{(\log(n+1))^\gamma}\left(1 - \frac{\gamma}{\log(n+1)}\right)\right\} \geq \exp\left\{\frac{1}{(\log(n+1))^\gamma}\right\}.
\end{aligned}
$$

■

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