General Nörlund transforms and power series methods

by

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1 Introduction and main result

Let us start by recalling the following summability methods:
We say a sequence (s_n) of complex numbers is summable to s by the

(i) **Cesàro-method** $s_n \to s(C_\alpha)$ for $\alpha > 0$ if

$$\frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} s_k \to s \quad (n \to \infty).$$

(ii) **Euler-method** $s_n \to s(E_\alpha)$ for $\alpha > 0$ if

$$\frac{1}{(\alpha+1)^n} \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} s_k \to s \quad (n \to \infty).$$

(iii) **Abel-method** $s_n \to s(A)$ if

$$(1-t) \sum_{n=0}^{\infty} s_n t^n \to s \quad (t \to 1-).$$

(iv) **Borel-method** $s_n \to s(B)$ if

$$e^{-t} \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n \to s \quad (t \to \infty).$$

The Cesàro- and Abel-method resp. the Euler- and Borel-method are known to be closely related, see [13,23,25]. Recall especially the following two classical results, see [13, Theorem 92; 15] resp. [1, Theorem 0; 16]

THEOREM A

(i) For $0 < \alpha \le \beta$ *:*

$s_n \to s(C_\alpha)$ *implies* $s_n \to s(C_\beta)$ *implies* $s_n \to s(A)$.
(Briefly: $(C_\alpha) \subseteq (C_\beta) \subseteq (A)$ *).*

(ii) For $s_n = \boldsymbol{O}(1)$ $s_n \to s(A)$ *implies* $s_n \to s(C_\alpha)$ *for all* $\alpha > 0$ *.*

THEOREM B

(i) For $0 < \alpha \le \beta$ *:* $(E_\alpha) \subseteq (E_\beta) \subseteq (B)$.

(ii) For $s_n = \boldsymbol{O}(1)$ $s_n \to s(B)$ *implies* $s_n \to s(E_\alpha)$ *for all* $\alpha > 0$.

Statement (i) of both theorems is an implication of Abelian character. It say, that the applicability of the Cesàro- resp. Euler-methods grows wider with increasing α, in the sense that the sets of summable sequences are increasing. For $\alpha \to \infty$ these sets tend in a certain sense to the set of (A) resp. (B) summable sequences.

Statement (ii) is of Tauberian character and really says, that for bounded sequences the class of Cesàro-methods and Abels method resp. the class of Euler-methods and Borels method are equivalent, briefly $(C_\alpha) \approx (A)$ resp. $(E_\alpha) \approx (B)$.

Discussion of the properties of power series methods, especially Tauberian theorems, i.e. implications from convergence of the power series method to convergence of some weaker summability methods or ordinary convergence, has a long history, see e.g. the books [13,23,25], and has recently found new attention in the papers [11,14,24] and [8,18,19,20].

In the latter papers general power series methods are discussed. These methods are defined in the following way.
Assume that (p_n) is a sequence of reals with

$$\text{(1.1)} \qquad \begin{cases} p_0 > 0, \quad p_n \ge 0, \quad n \in I\!N, \text{ such that the power series} \\ p(t) = \displaystyle\sum_{n=0}^{\infty} p_n t^n \quad \text{has radius of convergence } R > 0. \end{cases}$$

For complex sequences (s_n) with increments $a_n = s_n - s_{n-1}, n \ge 0, s_{-1} = 0$, we consider the *power series method* of summability (P) , where we say

$$\text{(1.2)} \qquad \begin{cases} s_n \to s(P), \quad \text{if } p_s(t) = \displaystyle\sum_{n=0}^{\infty} s_n p_n t^n \quad \text{converges for } |t| < R \\ \qquad\qquad \text{and if } \sigma_p(t) = \dfrac{p_s(t)}{p(t)} \to s \text{ as } t \to R- . \end{cases}$$

The power series methods contain the (J_p) - or Abel-type methods with $R = 1$ and Abel's method, $p_n = 1$, as a special case. The (B_p) - or Borel-type methods with $R = \infty$ and Borel's method, $p_n = 1/n!$, as a special case are similarly covered.

Since we can use $p_n R^n$ as weights in case $0 < R < \infty$, we only have to deal with the two cases $R = 1$ and $R = \infty$.
Furthermore we require the (P)-methods to be regular, i.e. $s_n \to s \ (n \to \infty)$ implies $s_n \to s(P)$. By Theorem 5 in [13], we have regularity if and only if

$$(1.3) \quad \begin{cases} (A) \quad P_n = \sum_{k=0}^{n} p_k \to \infty, \ \ (n \to \infty), \text{ in case } R = 1, \\[2ex] (B) \quad p(t) \text{ is not a polynomial, i.e.} \\[2ex] \qquad\ \ p_n \neq 0 \text{ for infinitely many } n \text{ in case } R = \infty. \end{cases}$$

During the discussion of general power series methods the sequence of maximal weights" (Δ_n) defined by

$$(1.4) \qquad \Delta_n = \inf_{0<t<R} p(t) t^{-n}.$$

played a decisive role, especially in formulating Tauberian conditions.
For a general discussion the weights p_n cannot be too irregular, so we have to impose a regularity condition.

$$(1.5) \qquad p_n \sim q_n = \exp\{-g(n)\}, \quad (n \to \infty),$$

with a real function g satisfying the conditions

$$(C) \quad \begin{cases} g \in C_2[x_0, \infty) \text{ for some } \quad x_0 \in I\!N, \\[2ex] g''(x) \text{ is positive and decreasing, with } \lim_{x\to\infty} g''(x) = 0, \text{ and} \\[2ex] G(x) := x^2 g''(x) \text{ is increasing on } [x_0, \infty) \end{cases}$$

Under (1.5) all 'smooth' sequences (p_n) are included with p_n growing faster than some positive power of n. So the Abel-method is not directly included, but we treat it as belonging to the limiting case $q_n = L(n)$, with $L(.)$ an increasing, slowly varying function, i.e $L \in SV$, see [4], §1.2 for the definition.

Under these conditions we have the relationship

$$\Delta_n = \phi(n)\, p_n \quad (n \to \infty), \tag{1.6}$$

with $\phi(t)$ a suitable, continuous function.

The aim of this note is to find matrix methods, which have similar properties related to general power series methods as our paradigms Cesàro- and Euler-methods have. We especially want to establish results similar to Theorems A and B for general power series methods. Results as in part (ii) are very useful in proving Tauberian theorems, since general Tauberian conditions always imply the boundedness of the sequence (s_n) in question, see [18, Theorem 1], and then one can work with the matrix method, which is easier to handle.

To this end we introduce the $(N, p^{*\alpha}, p)$ *-method of summability* for $\alpha \in I\!N$. These methods are defined in the following way:

Let (p_n) be a sequence of reals satisfying (1.1) and

$$p_n^{*\alpha} > 0, \quad n = 0, 1, 2, \ldots \quad \alpha \in I\!N - \{1\}, \tag{1.7}$$

where $p_n^{*\alpha}$ is defined by

$$p_n^{*1} := p_n, \quad n = 0, 1, 2, \ldots \quad \text{and } p_n^{*(\alpha+1)} := \sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k \tag{1.8}$$

Then we say $s_n \to s \ (N, p^{*\alpha}, p)$ if

$$\sigma_n^{\alpha} := \frac{1}{p_n^{*(\alpha+1)}} \sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k s_k \ \to s \ (n \to \infty). \tag{1.9}$$

Remark 1:
For $\alpha > 0$ a noninteger, we define $p_n^{*\alpha}$ as the n-th coefficient in the power series expansion of $p(t)^\alpha$ at 0, which is defined since $p_0 \neq 0$. We use the above definition, since we concentrate on the integer case only.

We can now state our main theorem

THEOREM 1

Let (p_n) be a sequence of reals satisfying (1.1) *and* (1.5), *then*

(i) *for* $1 \le \alpha \le \beta$, $\alpha, \beta \in I\!N$, *we have*

$$(N, p^{*\alpha}, p) \subseteq (N, p^{*\beta}, p) \subseteq (P).$$

(ii) $s_n = \boldsymbol{O}(1)$ *and* $s_n \to s\ (P)$ *imply* $s_n \to s\ (N, p^{*\alpha}, p)$ *for all* $\alpha \in I\!N$.

Remark 2

(i) Since $p_n \equiv 1$ (as the limiting case) resp. $p_n = 1/n! \sim \exp\{(n+\frac{1}{2})\log n - n + \frac{1}{2}\log(2\pi)\}$ by Stirling's formula in the Abel resp. Borel case Theorems A and B are included.

(ii) Under the condition $s_n = \boldsymbol{O}(1)$ the power series methods (P), with weight sequence (p_n), and (Q), with weight sequence (q_n), are equivalent, see [19] Lemma 2, and the same is true for the generalized Nörlund means (N, p, p) and (N, q, q).

(iii) The inclusions in Theorem 1 can be obtained under weaker conditions, see §2, Proposition 1.

(iv) The following methods are also covered, see [17,18,19,20] for further discussion and references

Abel–type

(i) (J_p) Methods with regularly varying weights (A_{R_α}):
$p_n = n^\alpha L(n)$, $\alpha > 0$, $L(.) \in SV$, $\Delta_n \sim cn$ with $\alpha = 0$, $L(t) \equiv 1$ as the limiting case being the classical Abel method (A).

(ii) Methods with rapidly increasing weights (A_p):
for example $p_n = \exp\{n^\alpha\}$, $0 < \alpha < 1$, $\Delta_n \sim cn^{1-\frac{\alpha}{2}} p_n$.

Borel–type

(iii) Generalized Borel-methods $(B(\alpha,\beta))$:
$p_n = \dfrac{\alpha}{\Gamma(\alpha n + \beta)}$, t replaced by t^α, $\alpha, \beta > 0$, $\Delta_n \sim c\sqrt{n} p_n$ with $\alpha = \beta = 1$ the classical Borel-method (B).

(iv) (B_p)–Methods:

for example $p_n = \exp\{-n^\beta\}$, $1 < \beta < 2$, $\Delta_n \sim cn^{1-\frac{\beta}{2}} p_n$.

We start out in §2 to discuss the relation of the $(N, p^{*\alpha}, p)$ - methods to other summability methods, especially the weighted mean methods and methods of moving averages. We also prove a slight generalization of Theorem 1(i). Furthermore we give an account of the historical context of the generalized Nörlund methods.
In §3 we concentrate on the case $\phi(t) = \boldsymbol{o}(t)$ and generalize a classical theorem on the connection between Cesàro- and Euler-methods. In §4 we complete the

proof of Theorem 1 and finally in §5 we give applications of the use of $(N, p^{*\alpha}, p)$-methods in probablitity theory and introduce some new families of summability methods comparable to the class of circle-methods, see [21].

2 Definitions and preliminary results

The first definition of the generalized Nörlund transform (N, a, b) was given by D. Borwein in [6] and is as follows:
Let (a_n) and (b_n) be sequences of real numbers such that

$$(a * b)_n = \sum_{k=0}^{n} a_{n-k} b_k \neq 0, \quad n = 0, 1, \ldots$$

Given a sequence (s_n) of complex numbers, its sequence (σ_n) of (N, a, b)-transforms is defined by

$$\sigma_n = \frac{1}{(a * b)_n} \sum_{k=0}^{n} a_{n-k} b_k s_k \tag{2.1}$$

and we say $s_n \to s(N, a, b)$, if $\sigma_n \to s \ (n \to \infty)$.

Using the Toeplitz theorem [13, Theorem 2] Borwein gave the necessary and sufficient conditions for the regularity of the (N, a, b)-methods,

$$\begin{aligned} &\text{(i)} \quad \sum_{k=0}^{n} |a_{n-k} b_k| = \boldsymbol{O}((a * b)_n) \\ &\text{(ii)} \quad \frac{a_{n-k} b_k}{(a * b)_n} \to 0 \ (n \to \infty) \text{ for each integer } k. \end{aligned} \tag{2.2}$$

A systematic discussion of the relation of the (N, a, b)-methods to weighted means $(N, \mathbf{1}, b)$ and Nörlund means $(N, a, \mathbf{1})$, where $\mathbf{1}$ stands for the sequence $(1, 1, \ldots)$, was given by Das in [9,10].
Although Borwein and Das observed that Cesàro- and Euler-methods are special cases of (N, a, b)-methods, none of them mentioned the possibility of generating them as $(N, p^{*\alpha}, p)$-methods.

We now start our investigation with a slight generalization of Theorem 1(i).

PROPOSITION 1
Let $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}$and (p_n) be a sequence of weights satisfying (1.1) *and* (1.7)*.*
(i) If (p_n) generates a regular (P) - method, then

$$(N, p^{*\alpha}, p) \subseteq (P).$$

(ii) If (p_n) generates regular $(N,p^{*\alpha},p)$-methods for all $\alpha \in I\!N$, then

$$(N,p^{*\alpha},p) \subseteq (N,p^{*\beta},p).$$

Proof:
To show the inclusion (i), we first show that each sequence $(p_n^{*\alpha})$ generates a regular (P)-method.
In case $R=1$, $p(t)^\alpha$ still has convergence radius $R \geq 1$ and since $(p*p)_n \geq p_0 p_n$, we have

$$\sum_{k=0}^{n} (p*p)_k \geq p_0 \sum_{k=0}^{n} p_k = p_0 P_n \to \infty \ (n \to \infty).$$

Therefore $R=1$ and the method is regular.

In case $R=\infty$ condition (1.7) implies the regularity.

Now let $s_n \to s \ (N,p^{*\alpha},p)$. By the regularity of all methods follows that

$$\sigma_p(t) = \frac{p_s(t)}{p(t)} = \frac{\sum_{n=0}^{\infty} p_n^{(\alpha+1)} \left(\frac{1}{p_n^{(\alpha+1)}} \sum_{k=0}^{n} p_{n-k}^{*\alpha} p_k s_k \right) t^n}{\sum_{n=0}^{\infty} p_n^{(\alpha+1)} t^n} \to s, \quad (t \to R-).$$

For the second inclusion we use the Toeplitz theorem to show that

$$A = (N,p^{*\beta},p)(N,p^{*\alpha},p)^{-1}$$

is regular. The inverse method $(N,p^{*\alpha},p)^{-1}$ can be obtained by using the power series $p(t) = \sum_{n=0}^{\infty} p_n t^n$. Since $p_0 \neq 0$, $p(t)^{-\alpha}$ has a power series expansion in some neighbourhood of 0, say $p(t)^{-\alpha} = c(t) = \sum_{n=0}^{\infty} c_n t^n$. Then

$$(N,p^{*\alpha},p)^{-1}_{(n,k)} = \begin{cases} \dfrac{c_{n-k} p_k^{*(\alpha+1)}}{p_n} & \text{for} \quad k \leq n \\ 0 & \phantom{\text{for}} \quad n < k. \end{cases}$$

For $k \leq n$ we hence have

$$\begin{aligned} A_{(n,k)} = a_{n,k} &= \frac{1}{p_n^{*(\beta+1)}} \sum_{\nu=k}^{n} p_{n-\nu}^{*\beta} p_\nu \frac{1}{p_\nu} c_{\nu-k} p_k^{*(\alpha+1)} \\ &= \frac{p_k^{*(\alpha+1)}}{p_n^{*(\beta+1)}} \sum_{\nu=k}^{n} p_{n-\nu}^{*\beta} c_{\nu-k} \\ &= \frac{p_k^{*(\alpha+1)} p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}}, \end{aligned}$$

where we used the coefficient formula for power series.

Since the $(N, p^{*\alpha}, p)$ - matrices are lower diagonal matrices so are the inverse matrices and therefore A, hence $a_{n,k} = 0$, if $k > n$.

Now $a_{n,k} \geq 0$ for all $n, k \geq 0$ and

$$\sum_{k=0}^{n} a_{n,k} = \frac{1}{p_n^{*(\beta+1)}} \sum_{k=0}^{n} p_k^{*(\alpha+1)} p_{n-k}^{*(\beta-\alpha)} = \frac{p_n^{*(\beta+1)}}{p_n^{*(\beta+1)}} = 1$$

by the coefficient formula. Therefore the only Toeplitz condition, which remains to be verified is that for each $k \in I\!N$

$$\frac{p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}} \to 0, \quad (n \to \infty).$$

By the regularity of the $(N, p^{*\gamma}, p)$-methods, $\gamma \in I\!N$, follows

$$\frac{p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}} = \frac{p_{n-k}^{*(\beta-\alpha)}}{p_{n-k}^{*(\beta-\alpha+1)}} \times \frac{p_{n-k}^{*(\beta-\alpha+1)}}{p_{n-k}^{*(\beta-\alpha+2)}} \times \ldots \times \frac{p_{n-k}^{*\beta}}{p_n^{*(\beta+1)}} \to 0, \quad (n \to \infty) \blacksquare$$

Remark 3:

(i) Parts of the proof can also be deduced using results of Das in [9, Theorem 1] and [10, Theorem 1].

(ii) Das also discussed in [10] the the Relations of (N, a, b) methods to each other. He particulary showed that if

$$\left.\begin{array}{c} p_{n+1} \geq p_n \quad \text{or} \\ \\ p_n \geq p_{n+1} \text{ and } P_n = \boldsymbol{O}((p * p)_n) \end{array}\right\} \text{ then } (N, \mathbf{1}, p) \subseteq (N, p, p).$$

Under the condition $p_{n+1}/p_n \leq p_{n+2}/p_{n+1}$ he also showed the reserve inclusion. In case $R = 1$ we can relax this condition.

LEMMA
Let (p_n) *be a non-increasing sequence of reals satisfying* (1.1) *with* $R = 1$ *. Then*

$$(N, p, p) \subseteq (N, \mathbf{1}, p)$$

Proof:

Let $p(t)^{-1} = c(t) = \sum_{n=0}^{\infty} c_n t^n$ and $C_n = \sum_{k=0}^{n} c_k$. Then $C_0 = c_0 > 0$ and by induction $C_n \geq 0$, since

$$0 = \sum_{k=0}^{n} c_k p_{n-k} = C_n p_0 + \sum_{k=0}^{n-1} C_k \underbrace{(p_{n-k} - p_{n-k-1})}_{\leq 0}.$$

Now one can use the Toeplitz theorem to show that

$$(N, \mathbf{1}, p)(N, p, p)^{-1}_{n,k} = \frac{(p * p)_k C_{n-k}}{P_n}$$

is regular. ■

While the $(N, \mathbf{1}, p)$ and the $(N, p, \mathbf{1})$ are obviously related to the (N, p, p)-methods their use in proofs of theorems on (N, p, p) means is limited to the case $t = \boldsymbol{O}(\phi(t))$. In case $\phi(t) = \boldsymbol{o}(t)$ the methods of moving averages are more helpful. These methods are very important in probability theory, see [3,17], and will be needed in the proof of Theorem 1(ii).

We say a sequence (s_n) of complex numbers is summable to s by the method of *moving averages* (R_ϕ), where $\phi(.)$ is a self-neglecting function, $\phi \in SN$, see [3,4 §2.11], briefly $s_n \to s(R_\phi)$ if

$$\text{(2.3)} \qquad \frac{1}{u\phi(n)} \sum_{n \leq k < n + u\phi(n)} s_k \to s, \quad \forall u > 0, \quad (n \to \infty).$$

These methods are related to the Cesàro-method by the following

PROPOSITION 2
The following statements are equivalent

(i) $\displaystyle \frac{1}{n+1} \sum_{k=0}^{n} (s_k + \epsilon_k) = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right)$ $(n \to \infty)$ *for some* $\epsilon_n \to 0$.

(ii) $s_n \to s(R_\phi)$.

Proof :
For $\phi(t) = t^\gamma,\ 0 < \gamma < 1$ this is Theorem 3 in [5]. The general case follows from Theorem 2 in [2], since this theorem is applicable also for $\phi \in$ SN. Compare also the proof of Theorem 1 in [3]. ■

We conclude the section by computing the asymptotics of

$$p_n^{*2} = (p * p)_n = \sum_{k=0}^{n} p_{n-k} p_k$$

in two important cases .

(1.5 A) $$p_n \sim n^\sigma L(n)$$

with $\sigma > -1, L(.)$ slowly varying.
This gives an Abel-type method of polynomial growth with convergence radius $R = 1$.
If we have $\lim_{x \to \infty} G(x) = \sigma > 0$ we are in the general case (1.5) with $\sigma > 0$, compare [8].

(1.5 B) $$p_n \sim \exp\{-g(n)\} \text{ with } \lim_{x \to \infty} G(x) = \infty\,.$$

These are the Abel-type methods with rapidly increasing weights and the Borel-type case. In this case the function $\phi(.)$ of (1.6) is selfneglecting.

A discussion of these conditions with further implications on the behaviour of the weights and the generated power series methods is given in [8,18,19,20].
Especially the asymptotic behaviour of $p_n t^n / p(t)$ has been studied in detail under these conditions. Since the methods used there are applicable for our problem, we only outline the proofs.

PROPOSITION 3

Let (p_n) *be a sequence of reals satisfying* (1.1) *and* (1.5) *then*

$$(p*p)_n \sim \begin{cases} n^{2\sigma+1}L(n)^2 B(\sigma+1,\sigma+1) & \text{(A)} \\ & \textit{in case} \\ \sqrt{\frac{\pi}{g''(n/2)}}\exp\{-2g(n/2)\} & \text{(B)} \end{cases}$$

($B(\sigma,\sigma)$ *denotes the Beta-Integral* $\int_0^1 t^{\sigma-1}(1-t)^{\sigma-1}\,dt$ *.)*

Proof:

Case (A): compare [14]

$$\begin{aligned}(p*p)_n &\sim \int_0^n (n-x)^\sigma x^\sigma L(n-x)L(x)\,dx \\ &= n^{2\sigma+1}L(n)^2\int_0^1 (1-x)^\sigma x^\sigma \frac{L(n(1-x))L(x)}{L(n)^2}\,dx \\ &\sim n^{2\sigma+1}L(n)^2 B(\sigma+1,\sigma+1) \ \ (n\to\infty).\end{aligned}$$

Case (B): Compare [8,19,20]

We use the so-called saddle point method. Define

$$\begin{aligned}\varphi(n,k)=\varphi(k) &= (-g(n-k)+g(n/2)) - g(k) + g(n/2) \\ &= -1/2(k-n/2)^2(g''(\xi_1)+g''(\xi_2))\end{aligned}$$

by Taylor's formula with $k \le \xi_2 \le n/2 \le \xi_1 \le n-k$.

We define now a function $\epsilon(x)$, compare [18,19,20], as

$$\epsilon(x) = x(x^2 g''(x))^{-\frac{1}{4}}.$$

Then we have

$$\frac{\epsilon(x)}{x} \to 0 \quad \text{and} \quad \epsilon(x)\sqrt{g''(x)} \to \infty \ (x\to\infty).$$

Furthermore the following basic inequality holds true

(2.4) $$\left|\frac{g(t)}{g(x)} - 1\right| \le \frac{\epsilon(x)}{x} \quad \text{for } |t-x| \le \epsilon(x).$$

Then using the properties of $\epsilon(.)$, especially (2.4), and the basic inequality

$|1-e^{-x}| \le x,\quad \text{for } 0 \le x$, we obtain

$$\frac{(p*p)_n}{\exp\{-g(n/2)\}} = 2\int_{\frac{n}{2}-\epsilon(\frac{n}{2})}^{\frac{n}{2}} \exp\{\varphi(n,x)\}\,dx + \boldsymbol{o}(1)$$

$$= \int_{\frac{n}{2}-\epsilon(\frac{n}{2})}^{\frac{n}{2}+\epsilon(\frac{n}{2})} \exp\{-g''(n/2)(x-n/2)^2\}\,dx$$

$$+2\int_{\frac{n}{2}-\epsilon(\frac{n}{2})}^{\frac{n}{2}} \exp\{-g''(n/2)(x-n/2)^2\}$$

$$\times\left\{\exp\left\{-\frac{1}{2}g''(n/2)(x-n/2)^2\left(\frac{g''(\xi_1)}{g''(n/2)}-1+\frac{g''(\xi_2)}{g''(n/2)}-1\right)\right\}-1\right\}\,dx + \boldsymbol{o}(1)$$

$$= \sqrt{\frac{\pi}{g''(n/2)}} + \boldsymbol{o}(1)\,.$$

■

Remark 4:

(i) The asymptotics show, that in the above cases the (N,p,p)- methods are regular, since $p_{n-k}/(p*p)_n \to 0 \ \ (n\to\infty) \ \ \forall k \in I\!N$

(ii) For higher order convolutions $p_n^{*\alpha}$ asymptotics can be obtained with the same methods. These asymptotics show the regularity of the $(N,p^{*\alpha},p)$- methods.

(iii) If we assume p_n/p_{n+1} to be nondecreasing we have

$p_{n-k}p_k \le p_{n-(k+1)}p_{k+1}$ if $k \le n/2$ and $p_{n-k}p_k \ge p_{n-(k+1)}p_{k+1}$ if $k \ge n/2$.

Therefore the maximal weight is

$$\frac{p_{n/2}p_{n/2}}{(p*p)_n},$$

which is asymptotically equal to p_n/Δ_n the maximal weight in the power series method (P) .

Observe that q_n/q_{n+1} is nondecreasing, where (q_n) is the the exact weight sequence in (1.5).

(iv) In case (1.5A) with nonincreasing (p_n) , i.e $-1<\sigma\le 0$, we have by Remark 3 and the Lemma, that $(N,\mathbf{1},p)$ summability is equivalent to (N,p,p) summability, if $cP_n \le (p*p)_n \le P_n$, with some constant $c>0$.

The asymptotics of Proposition 3 and and Lemmas 3.2 and 3.4 in [14] show that this can only be possible, if we are in the classical Cesàro-case.

3 The relation of Cesàros to the $(N, p^{*\alpha}, p)$-methods

We start by recalling a classical result of Knopp, see [13, Theorem 149]

THEOREM CE

If

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k = s + \boldsymbol{o}\left(1/\sqrt{n}\right), \quad (n \to \infty)$$

then $s_n \to s(E_\alpha)$ *for every* $\alpha > 0$.

For weights (p_n) satisfying condition (1.5B) with p_n/p_{n+1} nondecreasing, we can generalize this result.

Observe that in this cases we furthermore have $\phi(.) \in SN$ (with $\phi(.)$ as in (1.6)).

THEOREM 2

Let (p_n) *be a sequence of weights satisfying* (1.5B) *and*

(3.1) $\quad p_n/p_{n+1}$ *is nondecreasing. Then*

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right), \quad (n \to \infty)$$

implies $s_n \to s(N, p^{*\alpha}, p)$ *for every* $\alpha \in I\!N$.

Proof:

By the inclusion theorem we only have to show the implication for the (N, p, p)-method. Because of regularity and linearity we can suppose $s = 0$. Then

$$\sum_{k=0}^{n} s_k = \boldsymbol{o}\left(\phi(n)\right), \quad (n \to \infty). \tag{3.2}$$

Furthermore from the results in [8,19,20] we have under (1.5B) that $\phi(n) \sim \sqrt{(2\pi)/g''(n)}$, $(n \to \infty)$ and hence using Proposition 3

$$(p * p)_n \sim \sqrt{1/2}\phi(n/2)\exp\{-2g(n/2)\},\ (n \to \infty). \tag{3.3}$$

By Remark 4 (iii) and (3.1) we have therefore for the maximal weight in the

(N,p,p)-transform

$$(3.4) \qquad \frac{p_{n/2}p_{n/2}}{(p*p)_n} \sim \frac{\sqrt{2}}{\phi(n/2)},\ (n\to\infty).$$

Now let $\epsilon>0$ be given, we can find a $N\in I\!N$, such that for $n\geq l\geq m\geq N$

$$(3.5) \qquad \left|\sum_{k=m}^{l} s_k\right| \leq \epsilon\phi(l)\leq \epsilon\phi(n),$$

using also the monotonicity of $\phi(.)$. We split the (N,p,p)-transform in three parts, namely

$$\sigma_n=\frac{1}{p_n^{*2}}\sum_{k=0}^{n} p_{n-k}p_k s_k=\frac{1}{p_n^{*2}}\left(\sum_{k=0}^{N}+\sum_{k=N+1}^{n/2-1}+\sum_{k=n/2}^{n} p_{n-k}p_k s_k\right)=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3 .$$

By Abel's Partial Summation we get

$$|\textstyle\sum_2| \leq \left|\displaystyle\sum_{k=N+1}^{n/2-1} s_k\right|\frac{p_{n/2}^2}{p_n^{*2}}+\frac{1}{p_n^{*2}}\sum_{k=N+1}^{n/2-1}\left|\sum_{\nu=N+1}^{k} s_\nu\right| |p_{n-k}p_k-p_{n-k-1}p_{k+1}|$$

We now use (3.3) and (3.5) and the monotonicity of $p_{n-k}p_k$ by (3.1) to find

$$|\textstyle\sum_2| \leq \epsilon+\epsilon\displaystyle\frac{\phi(n/2)}{p_n^{*2}}(p_{n/2}^2-p_{n-N}p_N)\leq c_2\epsilon.$$

With a similar argument

$$|\textstyle\sum_3|\leq c_3\epsilon$$

follows. Since N is fixed and $\dfrac{p_{n-k}p_k}{p_n^{*2}}\to 0,\ (n\to\infty),\ \forall k$ it follows that

$$|\textstyle\sum_1|\leq c_1\epsilon$$

Since c_1,c_2,c_3 are constants the proof is complete. ∎

Corollary
With (p_n) and $\phi(.)$ as in Theorem 2 $s_n\to(R_\phi)$ implies $s_n\to s(N,p^{*\alpha},p)$ for all $\alpha\in I\!N$.

Proof:
With Proposition 2 $s_n \to (R_\phi)$ implies

$$\frac{1}{n+1}\sum_{k=0}^{n}(s_k+\epsilon_k) = s + \boldsymbol{o}\left(\frac{\phi(n)}{n}\right), \quad (n\to\infty)$$

with some sequence $\epsilon_n \to 0,\ (n\to\infty)$. Hence we obtain by the Theorem $(s_n+\epsilon_n) \to s(N,p^{*\alpha},p)$. Since $(N,p^{*\alpha},p)$ is a linear and regular summability method $s_n \to s(N,p^{*\alpha},p)$ follows. ■

Remark 5:

(i) The result of the Corollary can also be obtained by direct computation. One has to use the asymptotic properties of the (N,p,p) transform and the techniques in [17].

(ii) Also by the inclusion theorem we have $(N,p^{*\alpha},p) \subseteq (P)$. Therefore under the conditions of Theorem 2 (P)-summability is implied.

(iii) Under the condition $s_n = \boldsymbol{O}(1)$ we can omit the assumption of monotonicity (3.1).

4 Proof of theorem 1

Proof:
The proof of part(i) follows from Proposition 1 and Remark 4(ii).
To prove part(ii) we only have to show $s_n \to s(N,p,p)$. The other implications follow by inclusion.
We distinguish two cases:
(i) If $G(x) \to \sigma > 0$, we have the case of polynomial weights. We include the limiting case $\sigma = 0$, i.e $p_n \sim L(n)$ with $L \in SV$ and nondecreasing.
Furthermore since $s_n = \boldsymbol{O}(1)$ we can switch to the asymptotic weights $q_n = n^\sigma L(n)$, with $\sigma \geq 0,\ L \in SV$ and nondecreasing.
We hence have $s_n \to s(Q)$, where (Q) means the power series method with weight sequence (q_n).
In this situation we use a general theorem for Laplace transforms by Stadtmüller/ Trautner, see [24], to get

$$s_n \to s(N,\mathbf{1},q).$$

Now, since (q_n) increases we have $s_n \to s(N,q,q)$ by Remark 3.
We use once again the condition $s_n = \boldsymbol{O}(1)$ to get $s_n \to s(N,p,p)$.

(ii) (p_n) as in (1.5B), hence $G(x) \to \infty\ (x\to\infty)$ and the asymptotic weight-sequence is $q_n = e^{-g(n)}$. Now we use Theorem 1 in Moh [22].

Define $s(u) = s_{[u]}$ if $u \geq 0$ and $s(u) = 0$ if $u < 0$. As in Kiesel/Stadtmüller [18] we find that under $s_n = \boldsymbol{O}(1)$ $s_n \to s(P)$ implies

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} K\left(\frac{x-t}{\phi(x)}\right) s(t) \frac{dt}{\phi(x)} = s \tag{4.1}$$

with $K(x) = 1/\sqrt{2\pi}\exp\{-x^2/2\}$. The conditions of Theorem 1 of Moh, i.e. $K(x) \in L^1(-\infty,\infty)$, the Fourier transform of $K \neq 0$ for any real argument and $\phi(.) \in SN$, are satisfied.
It follows that if we define

$$H(x) = \begin{cases} \dfrac{1}{\epsilon}, & \text{if} \quad x \in (-\epsilon, 0), \\ 0, & \phantom{\text{if}} \quad x \notin (-\epsilon, 0) \end{cases}$$

that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} H\left(\frac{x-t}{\phi(x)}\right) s(t) \frac{dt}{\phi(x)} = \lim_{x\to\infty} \frac{1}{\epsilon\phi(x)} \sum_{x\leq k<x+\epsilon\phi(x)} s_k = s. \tag{4.2}$$

This means $s_n \to s(R_\phi)$. By the Corollary to Theorem 2 and the monotonicity of q_n/q_{n+1} $s_n \to s(N,q,q)$ follows. Using again that $s_n = \boldsymbol{O}(1)$ we can conclude that $s_n \to s(N,p,p)$. ■

Remark 6
We consider the case $\phi(t) = \boldsymbol{o}(t)$ only.

(i) If we use $s_n - s_{n-1} = \boldsymbol{O}(\phi(n)/n)$ as additional condition in the setting of Theorem 1 the convergence of s_n to s follows. To see that let ϵ tend to 0 in (4.2).

(ii) Under $s_n = \boldsymbol{O}(1)$ we have the following equivalences: $(R_\phi) \Leftrightarrow (N, p^{*\alpha}, p) \Leftrightarrow (P)$, compare Bingham [1] for the Euler-Borel-case.

(iii) It is also possible to obtain a proof of part (ii) by direct computation using the asymptotic properties of the (N,p,p) and (P)-transforms.

5 Applications and complements

(i) The (N,p,p)-methods are the natural methods to be used as general matrix summability methods in the following characterisation of moment conditions by summability methods, see [17] Theorem.

Let $X, X_0, X_1, \ldots$ be i.i.d. random variables and (P) a power-series method with weight sequence (p_n) satisfying (1.5). Then the following statements are equivalent.

(i) $E(\phi^{-1}(|X|)) < \infty,\ E(X) = \mu;$

(ii) $X_n \to \mu(N, p^{*\alpha}, p)\ a.s.\ \forall \alpha \in I\!N;$

(iii) $X_n \to \mu(P) a.s.$

In the proof for $\phi(t) \approx t$ the (C_α)-methods were used and for $\phi(t) = \boldsymbol{o}(t)$ the Riesz-methods $(R, \exp\{\Phi(n)\}, 1)$ with $\Phi(x) = \int_1^x dt/\phi(t)$.
But $(R, \exp\{\Phi(n)\}, 1) \Leftrightarrow (R_\phi)$ by Lemma 3 in [17] and (R_ϕ) implies (N, p, p).

(ii) One can also use the (N, p, p)-methods to obtain new examples for Theorem 1 in Borwein [7], where weakly multipicative sequences of random variables are studied.

(iii) In case $\phi(t) = \boldsymbol{o}(t)$ and (1.5B) or (1.5C) one can attempt to generalize the well-known Taylor (T) and Meyer-König (S) - methods of summability. Together with Euler-, Borel- and Valiron-method these methods build the family of circle methods, see [21].
One observes that $(T_\alpha) = (E_\alpha)^t$ and (S_α) is obtained from (T_α) by omitting the leading zeros in the (T_α) methods. Using this procedure starting with an (N, p, p)-method a (T, p, p)-method and a (S, p, p)-method can be formally obtained.
Without being explicitly stated a general Valiron method is already in long use (see the definition in (4.1) and [18]).

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