On scales of summability methods

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Abstract. In this paper we consider general Nörlund means $(N_{\alpha p}), \alpha > -1$ and power series methods (J_p) and the iteration product of two such methods. A particular case is that of the Cesàro methods (C_α) with corresponding power series method (A) , i.e. Abel's method. We obtain generalizations of inclusion and tauberian and convexity theorems, which are well known for the Cesàro-Abel case, for a rich class of methods of the above type.

1 Introduction and main result

Let us start by recalling the following summability methods, see [8,15,20]:
We say a sequence (s_n) of complex numbers is summable to s by the

(i) *Cesàro-method* $s_n \to s(C_\alpha)$ for $\alpha > -1$ if

$$\sigma_n^\alpha := \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} s_k \to s \quad (n \to \infty),$$

(ii) *Abel-method* $s_n \to s(A)$ if

$$\sigma(x) := (1-x) \sum_{n=0}^{\infty} s_n x^n \to s \quad (x \to 1-).$$

The Cesàro- and Abel-method are known to be closely related. Recall especially the following classical result, see [8, Theorem 92; 20, §53 I and §55 II, IV].

Theorem HL.
(i) For $-1 < \alpha \leq \beta$ *:*

$s_n \to s(C_\alpha)$ *implies* $s_n \to s(C_\beta)$ *implies* $s_n \to s(A)$.
(Briefly: $(C_\alpha) \subseteq (C_\beta) \subseteq (A)$ *).*

(ii) $s_n \to s(A)$ *and* $s_n = \boldsymbol{O}(1)$ *imply* $s_n \to s(C_\alpha)$ *for all* $\alpha > 0$.

Statement (i) is an implication of *Abelian* character. It say, that the applicability of the Cesàro methods grows wider with increasing α, in the sense that the sets of summable sequences are increasing. For $\alpha \to \infty$ the set of (A)-summable sequences can be viewed as a limit set, although there are (A)-summable sequences which are not summable by any (C_α)-method.

Statement (ii) is of *Tauberian* character and really says that for bounded sequences the class of Cesàro methods and Abel's method are equivalent (briefly $(C_\alpha) \approx (A)$), if one imposes a *Tauberian condition* on the sequence (s_n) (in this case boundedness).

Theorem HL can be generalized to a so-called convexity theorem [1; 8 Theorem 70; 20 §53 V, §55]

Theorem A. *For* $-1 < \alpha < \beta \leq \gamma$ *:*

(i) $s_n \to s(A)$ *and* $s_n = \boldsymbol{O}(1)\,(C_\alpha)$ *(i.e.* $\sigma_n^\alpha = \boldsymbol{O}(1)$ *) imply* $s_n \to s(C_\beta)$.

(ii) $s_n \to s(C_\gamma)$ *and* $s_n = \boldsymbol{O}(1)\,(C_\alpha)$ *imply* $s_n \to s(C_\beta)$.

Using theorem HL we see that (ii) is an corollary of (i). In proving tauberian theorems it is also very useful to consider the iteration product of Cesàro and Abel methods [20, §55 V]. It is defined as follows, see [4,9]:

We say a sequence (s_n) of complex numbers is summable to s by the *iteration product* (AC_α) $s_n \to s(A,\alpha)$ for $\alpha > -1$ if

$$\sigma_n^\alpha \to s(A), \quad \text{i.e. } (1-x)\sum_{n=0}^{\infty} \sigma_n^\alpha x^n \to s \quad (x \to 1-).$$

Observe that $(A,0)$-summability is ordinary Abel-summability.
(For summability methods (M) and (N) we always use the term iteration product (MN) meaning the (M) method applied to the (N) transform.)
We have [9, Theorems 2.2, 2.3]

Theorem J. *Let* $-1 < \alpha \leq \beta, -1 < \gamma$. *Then*

(i) $s_n \to s(A,\alpha)$ *implies* $s_n \to s(A,\beta)$.

(ii) $s_n \to s(C_\gamma)$ *implies* $s_n \to s(A,\alpha)$.

The aim of this paper is to discuss similar relations for a general class of power series methods and related matrix methods of summability. We consider general *power series methods* (J_p) defined in the following way [3,6,13,18].
Assume that (p_n) is a sequence of reals with

$$(1.1)\qquad \begin{cases} p_0 > 0, \quad p_n \geq 0, \quad n \in I\!N, \text{ such that the power series} \\ p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{has radius of convergence } R = 1. \end{cases}$$

For complex sequences (s_n) with increments $a_n = s_n - s_{n-1}, n \geq 0, s_{-1} = 0$, we consider the *power series method* of summability (J_p), where we say

$$(1.2)\qquad \begin{cases} s_n \to s(J_p), \quad \text{if } p_s(x) = \sum_{n=0}^{\infty} s_n p_n x^n \quad \text{converges for } |x| < 1 \\ \qquad\qquad\qquad \text{and if } \sigma_p(x) = \dfrac{p_s(x)}{p(x)} \to s \text{ as } x \to 1-. \end{cases}$$

The power series methods (J_p) contain Abel's method, $p_n \equiv 1$, as a special case. We require the (J_p)-methods to be regular, i.e. $s_n \to s\ (n \to \infty)$ implies $s_n \to s(J_p)$. By Theorem 5 in [8], we have regularity if and only if

$$(1.3)\qquad P_n := \sum_{k=0}^{n} p_k \to \infty, \ (n \to \infty),$$

Without further mention we always assume (1.1) and (1.3).

For a general discussion the weights p_n cannot be too irregular, so we have to impose a regularity condition,

$$(1.4)\qquad np_n = \boldsymbol{O}(P_n), \quad (n \to \infty).$$

This includes especially the case of regularly varying weights $p_n = n^\sigma L(n)$, with $\sigma \geq -1$ and $L(.)$ an increasing, slowly varying function, i.e $L \in SV$, see [2], §1.2 for the definition.

As matrix methods, which have similar properties related to general power series methods as our paradigm Cesàro method we use, see [5,7,11,12], *generalized Nörlund methods* (N, a, b).

The first definition of the generalized Nörlund transform (N, a, b) was given by D.Borwein in [5] and is as follows:

Let (a_n) and (b_n) be sequences of real numbers such that

$$(a*b)_n := \sum_{k=0}^{n} a_{n-k}b_k \neq 0, \quad n = 0, 1, \ldots$$

Given a sequence (s_n) of complex numbers, its sequence $(\sigma_n^{a,b})$ of (N, a, b)-transforms is defined by

$$\sigma_n^{a,b} := \frac{1}{(a*b)_n} \sum_{k=0}^{n} a_{n-k}b_k s_k \tag{1.5}$$

and we say $s_n \to s(N, a, b)$, if $\sigma_n^{a,b} \to s \ (n \to \infty)$.

Using the Toeplitz theorem [8, Theorem 2] D.Borwein gave the necessary and sufficient conditions for the regularity of the (N, a, b)-methods,

$$\text{(1.6)} \qquad \begin{array}{ll} \text{(i)} & \displaystyle\sum_{k=0}^{n} |a_{n-k}b_k| = \boldsymbol{O}((a*b)_n) \quad (n \to \infty), \\ \text{(ii)} & \displaystyle\frac{a_{n-k}b_k}{(a*b)_n} \to 0 \quad (n \to \infty) \text{ for each integer } k. \end{array}$$

We observe that the (C_α)-methods are special general Nörlund methods $(N, \epsilon^{\alpha-1}, \mathbf{1})$ obtained by using the sequences

$$\epsilon^\alpha = (\epsilon_n^\alpha) := \left(\binom{n+\alpha}{n} \right)$$

and $\mathbf{1}$, where $\mathbf{1}$ stands for the sequence $(1, 1, \ldots)$.

Let $\alpha > -1$. We consider the general Nörlund means $(N, \epsilon^{\alpha-1}, p)$, in short notation $(N_{\alpha p})$. So (C_α) is included as $(N_{\alpha \mathbf{1}})$. We also use the notations:

$$p_n^\alpha := (\epsilon^{\alpha-1} * p)_n \quad \text{and } \sigma_n^{\alpha,p} := \sigma_n^{\epsilon^{\alpha-1},p}$$

for the $(N_{\alpha p})$ transform.

We define an *iteration product method* (J_p, α), $\alpha > -1$, where we say

$$s_n \to a(J_p, \alpha) \quad \text{if} \quad \sigma_n^{\alpha,p} \to s(J_p).$$

Our main results are

Theorem 1. *Let* (p_n) *satisfy* (1.4), *then*
$s_n \to s(J_p)$ *and* $s_n = \boldsymbol{O}(1)$ *imply* $s_n \to s(N_{\alpha p})$ *for all* $\alpha > 0$.

Theorem 2. *Let* (p_n) *satisfy* (1.4). *For* $-1 < \alpha < \beta$; $s_n \to s(J_p)$ *and* $s_n = \boldsymbol{O}(1)\,(N_{\alpha p})$ *imply* $s_n \to s(N_{\beta p})$.

Theorem 3. *Let* $p_n \sim q_n$, $(n \to \infty)$, *i.e.* $p_n/q_n \to c > 0$ $(n \to \infty)$, *with a sequence* (q_n) *that satisfies* (1.4) *and*

$$\text{(1.7)} \qquad \textit{for each } \alpha \geq 0 \quad \frac{q_n}{q_n^\alpha} \quad \textit{is totally monotone}$$

(see §2 for the definition). Then for bounded sequences (s_n) *we have for* $0 < \alpha, \beta$

$$(N_{\alpha p}) \approx (J_p, \beta).$$

We now start out in §2 to give the necessary definitions and a few preliminary results. In §3 we supply the proofs and conclude in §4 with a few remarks on related results and geralizations.

2 Definitions and preliminary results

For $\alpha, \beta \in \mathbb{R}$ the following identities are straightforward:

$$\text{(2.1)} \qquad p_n^{\alpha+\beta} = \sum_{k=0}^{n} \epsilon_{n-k}^{\alpha-1} p_k^\beta, \quad \text{especially} \quad p_n^{\alpha+1} = P_n^\alpha = \sum_{k=0}^{n} p_k^\alpha.$$

Remark 1. Using these identities we get for sequences (p_n) satisfying (1.4)

(i) $P_n^\alpha \to \infty$ for $\alpha \geq 0$.

(ii) $\epsilon_n^{\beta-\alpha-1} = \boldsymbol{o}\left(p_n^\beta\right)$, $n \to \infty$ for $0 \leq \alpha < \beta$.

(iii) (p_n^α) satisfies (1.4) for each $\alpha \geq 0$.

(iv) $\dfrac{p_n^\alpha}{P_n^\alpha} = \boldsymbol{O}\left(\dfrac{p_n^{\alpha+1}}{P_n^{\alpha+1}}\right)$ for $\alpha \geq 0$.

Now we have the following *inclusion property*

Proposition 1. *For* $-1 < \alpha \le \beta$ *we have* $(N_{\alpha p}) \subseteq (N_{\beta p}) \subseteq (J_p)$.

The propositon is proved via the two following lemmas.

Lemma 1. *If* $((a*p)_n)$ *satisfies* (1.1) *and* (1.3), *then* $(N,a,p) \subseteq (J_p)$.

P r o o f: Let $s_n \to s\ (N,a,p)$. By the regularity of all methods follows

$$\sigma_p(x) = \frac{p_s(x)}{p(x)} = \frac{\sum_{n=0}^{\infty}(a*p)_n \left(\frac{1}{(a*p)_n}\sum_{k=0}^{n} a_{n-k}p_k s_k\right) x^n}{\sum_{n=0}^{\infty}(a*p)_n x^n} \to s, \quad (x \to 1-).$$

Remark 2. The above proof shows that $(J_p) \approx (J_p^{\alpha}, \alpha)$ for each $\alpha \ge 0$.

The $(N_{\alpha p})$ methods satisfy a *product formula.*

Lemma 2. *For* $-1 < \alpha \le \beta$ *we have* $(N_{\beta p}) = (N, \epsilon^{\beta-\alpha-1}, p^{\alpha})(N_{\alpha p})$.

P r o o f: This follows from the identity

$$\sum_{k=0}^{n} \epsilon_{n-k}^{\beta-\alpha-1} \sum_{\nu=0}^{k} \epsilon_{k-\nu}^{\alpha-1} p_\nu s_\nu = \sum_{\nu=0}^{n} p_\nu s_\nu \sum_{k=\nu}^{n} \epsilon_{k-\nu}^{\alpha-1}\epsilon_{n-k}^{\beta-\alpha-1} = \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\beta-1} p_\nu s_\nu .$$

P r o o f of Proposition 1: By Remark 1 $(N, \epsilon^{\beta-\alpha-1}, p^{\alpha})$ (use (1.6)) and $(J_{p^{\alpha}})$ are regular, so the Proposition follows from Lemmas 1 and 2.

As a further elementary result we compute the sequence-to-series form of the $(N_{\alpha p})$-transforms.

Lemma 3. *For* $\alpha > -1,\ n \in I\!N$ *we have* $\sigma_n^{\alpha,p} - \sigma_{n-1}^{\alpha,p} = \dfrac{p_n^{\alpha-1}}{p_n^{\alpha}} \left(\sigma_n^{(\alpha-1),p} - \sigma_{n-1}^{\alpha,p} \right).$

P r o o f:
Using the well know identities for the binomial coefficients and the identities (2.1) we obtain:

$$\begin{aligned}
\sigma_n^{\alpha,p} - \sigma_{n-1}^{\alpha,p} &= \frac{p_n s_n}{p_n^{\alpha}} + \frac{1}{p_n^{\alpha} p_{n-1}^{\alpha}} \sum_{k=0}^{n-1} p_k s_k \left\{ \epsilon_{n-k}^{\alpha-1} p_{n-1}^{\alpha} - \epsilon_{n-1-k}^{\alpha-1} p_n^{\alpha} \right\} \\
&= \frac{p_n s_n}{p_n^{\alpha}} + \frac{1}{p_n^{\alpha} p_{n-1}^{\alpha}} \sum_{k=0}^{n-1} p_k s_k \left\{ p_{n-1}^{\alpha} (\epsilon_{n-k}^{\alpha-1} - \epsilon_{n-1-k}^{\alpha-1}) - \epsilon_{n-1-k}^{\alpha-1} p_n^{\alpha-1} \right\} \\
&= \frac{p_n s_n}{p_n^{\alpha}} + \frac{1}{p_n^{\alpha}} \sum_{k=0}^{n-1} \epsilon_{n-k}^{\alpha-2} p_k s_k + \frac{p_n^{\alpha-1}}{p_n^{\alpha} p_{n-1}^{\alpha}} \sum_{k=0}^{n-1} \epsilon_{n-1-k}^{\alpha-1} p_k s_k \\
&= \frac{p_n^{\alpha-1}}{p_n^{\alpha}} \left\{ \frac{1}{p_n^{\alpha-1}} \sum_{k=0}^{n} \epsilon_{n-k}^{\alpha-2} p_k s_k - \frac{1}{p_{n-1}^{\alpha}} \sum_{k=0}^{n-1} \epsilon_{n-1-k}^{\alpha-1} p_k s_k \right\}.
\end{aligned}$$

An immediate consequence is

Corollary 1. *If* $s_n = \boldsymbol{O}(1)$ *and* $\alpha \geq 1$ *then* $\left| \sigma_n^{\alpha,p} - \sigma_{n-1}^{\alpha,p} \right| = \boldsymbol{O} \left(\dfrac{p_n^{\alpha-1}}{p_n^{\alpha}} \right).$

To discuss the (J_p, α)-methods we have to impose a further regularity condition. We need

Definition. A sequence (μ_n) is *totally monotone*, if there is a nondecreasing function $\chi : [0,1] \to I\!R$ such that

$$\mu_n = \int_0^1 t^n d\chi(t).$$

We then have

Proposition 2. *Fix* $\alpha \geq 0$. *If* p_n / p_n^{α} *is totally monotone, then* $(J_p) \subseteq (J_p, \alpha)$.

P r o o f: By Remark 2 we have $(J_p) \subseteq (J_{p^{\alpha}}, \alpha)$. Under the above condition the proposition follows from a theorem of D.Borwein in [3], see also [6,12].

We now state for convenience a few known results, which will be needed in the proofs. We start with a version of R.Sinha's Theorem [16].

Theorem S. *Let* (p_n) *satisfy* (1.4), *then for* $-1 < \alpha < \beta \leq \gamma$

$$s_n \to s(N_{\gamma p}) \quad \textit{and} \quad s_n = \boldsymbol{O}\,(1)\,(N_{\alpha p}) \quad \textit{imply} \quad s_n \to s(N_{\beta p}).$$

Under our conditions an alterative p r o o f of this theorem using Vitali's Theorem [19, p.168] can be given, compare W.B.Jurkat [10].
Consider $\sigma_n^{z,p}$ as an analytic function of the complex variable $z = s + it$ in the rectangle $R_\delta := [\alpha + \delta \leq s \leq \gamma + 1, |t| \leq \delta]$ with a fixed, sufficiently small $\delta > 0$. Our assumptions imply that $\sigma_n^{z,p}$ is uniformly bounded in R_δ and by Vitali's Theorem the convergence of $\sigma_n^{z,p}$ for $\gamma \leq s \leq \gamma + 1$ implies convergence for each inner point of R_δ.

The following two theorems are from H.Tietz/R.Trautner [18].

Theorem ST. *Let* (p_n) *satisfy* (1.4), *then* $s_n = \boldsymbol{O}\,(1)$ *is a tauberian condition of type* $(J_p) \to (M_p)$, *i.e.* $s_n \to s(J_p)$ *and* $s_n = \boldsymbol{O}\,(1)$ *imply* $s_n \to s(M_p)$.

Theorem TT. *Let* (p_n) *satisfy* (1.4), *then* $a_n = \boldsymbol{O}\left(\frac{p_n}{P_n}\right)$ *is a tauberian condition of type* (J_p) *to convergence.*

3 Proofs

First p r o o f of Theorem 1.
By Theorem ST we have $s_n \to s(J_p)$ and $s_n = \boldsymbol{O}\,(1)$ imply $s_n \to s(M_p)$. Since (M_p) convergence is just (N_{1p}) convergence, the inclusion property gives $s_n \to s(N_{\alpha p})$ for $\alpha \geq 1$.
For $0 < \alpha < 1$ we choose a $0 < \delta < \alpha$ and then $s_n = \boldsymbol{O}\,(1)$ implies $s_n = \boldsymbol{O}\,(1)\,(N_{\delta p})$. Using Theorem S we obtain $s_n \to s(N_{\alpha p})$.

Second p r o o f of Theorem 1.
We start with a fixed $\beta > 1$. By Remark 2 we see that $s_n \to s(J_{p^\beta}, \beta)$. Using

Corollary 1 and Remark 1 it follows that under $s_n = \boldsymbol{O}(1)$ the sequence $(\sigma_n^{\beta,p})$ satiesfies the tauberian conditions of Theorem TT. Hence we have $s_n \to s(N_{\beta p})$. Using the inclusion property and Theorem S as in the first proof the result follows.

Let p_n^{*k} be the $n-$ th coefficient of the power series $p(x)^k$, $k \in \mathbb{N}$. As a corollary we obtain a result, which was proved in a different setting in [11].

Corollary 2. *If* $p_n \sim n^\rho L(n)$ *with* $\rho > -1$ *and* $L(.)$ *a slowly varying function, then* $s_n \to s(J_p)$ *and* $s_n = \boldsymbol{O}(1)$ *imply* $s_n \to s(N, p^{*k}, p)$ *for each* $k \in \mathbb{N}$ *.*

P r o o f: Theorem 1 gives $s_n \to s(N_{\alpha p})$ for each $\alpha > 0$. For α sufficiently small we use [7, Theorem 1 resp. Theorem 3] to get $(N_{\alpha p}) \subseteq (N, p, p)$. The inclusion property [11, Proposition 1] for the (N, p^{*k}, p)-methods then gives the result.

First p r o o f of Theorem 2.
Using again Remark 2 we get $s_n \to s(J_{p^\alpha}, \alpha)$. Since $\sigma_n^{\alpha,p} = \boldsymbol{O}(1)$ we obtain by Theorem ST $\sigma_n^{\alpha,p} \to s(M_{p^\alpha})$. By the product formula this is $s_n \to s(N_{(\alpha+1)p})$. By inclusion and Sinha's Theorem the result follows.

Second p r o o f of Theorem 2.
Let $\alpha < \beta$. As above we have $s_n \to s(J_{p^{\beta+1}}, \beta+1)$. Using Lemma 2 we see that this means $\sigma_n^{\alpha,p} \to s(J_{p^{\beta+1}})(N_{(\beta+1-\alpha)p^\alpha})$. Write $\tilde{\sigma}_n^{(\beta+1-\alpha),p^\alpha}$ for the $(N_{(\beta+1-\alpha)p^\alpha})$-transform of $(\sigma_n^{\alpha,p})$. Since $\sigma_n^{\alpha,p} = \boldsymbol{O}(1)$ it follows by Corollary 1 and Remark 1 (iv) that

$$\tilde{\sigma}_n^{(\beta+1-\alpha),p^\alpha} - \tilde{\sigma}_{n-1}^{(\beta+1-\alpha),p^\alpha} = \boldsymbol{O}\left(\frac{p_n^{\beta+1}}{P_n^{\beta+1}}\right).$$

Hence $(\tilde{\sigma}_n^{(\beta+1-\alpha),p^\alpha})$ satisfies the tauberian conditions of Theorem TT, and we obtain $\tilde{\sigma}_n^{(\beta+1-\alpha),p^\alpha} \to s$. By the product formula this is $s_n \to s(N_{(\beta+1)p})$, and the inclusion property and Theorem S give the desired result.

P r o o f of Theorem 3.
In view of Theorem 1 and Proposition 1 it is only left to prove that for bounded sequences (s_n) $(J_p, \beta) \approx (J_p)$ for each $\beta > 0$.
We start by proving that $s_n = \boldsymbol{O}(1)$ is a tauberian condition for $(J_p, \beta) \to (J_p)$ for each $\beta > 0$.

To this end we show the stronger proposition that $s_n = \boldsymbol{O}(1)$ is a tauberian condition for $(J_p, \beta) \to (N_{\gamma p})$ for each $\gamma > 0$.
Since $s_n = \boldsymbol{O}(1)$ we have $\sigma_n^{\beta,p} = \boldsymbol{O}(1)$ and we can switch to the asymptotic weights (compare [13, Lemma 2]) this means $\sigma_n^{\beta,p} \to s\ (J_q)$. By Theorem ST $\sigma_n^{\beta,p} \to s(M_q)$. Now (1.7) implies that q_n^β / q_n is nondecreasing and by using (1.4) we get

$$\frac{q_n^\beta Q_n}{Q_n^\beta q_n} = \boldsymbol{O}(1).$$

By [8, Theorem 14] these conditions imply that $\sigma_n^{\beta,p} \to s(M_{q^\beta})$. Now again the boundedness of $\sigma_n^{\beta,p}$ allows us to switch back to the weights (p_n) and obtain by the product property $s_n \to s(N_{(\beta+1)p})$. Since under $s_n = \boldsymbol{O}(1)$ all $(N_{\gamma p})$-methods are equivalent for $\gamma > 0$ the desired result follows.
The inclusion $(J_p) \subseteq (J_p, \beta)$ for each $\beta > 0$ follows by using the asymptotic weights, argue as above, and by Proposition 2.

4 Concluding Remarks

(i) Sinha's original theorem is more general and reads as follows:
Let (q_n) be a sequence of constants such that for each $\xi > -1$ there exist positive constants H_1, H_2 (which may depend on ξ but not on n) with

$$H_1 n^\xi \leq (p^\xi * q)_n / (p * q)_n \leq H_2 n^\xi.$$

Then if $-1 < \alpha < \beta \leq \gamma$,
$s_n \to s(N, p^\gamma, q)$ and $s_n = \boldsymbol{O}(1)\,(N, p^\alpha, q)$ imply $s_n \to s(N, p^\beta, q)$.

If (q_n) satisfies the conditions (1.1) and (1.3) and

$$p_n^\xi = \boldsymbol{o}\left(n^\xi (p * q)_n\right) \quad (n \to \infty),$$

for each $\xi \geq 0$, then one can prove theorems similar to Theorems 1-3 (using the first method of proof, since a sequence-to-series form is not accessible in closed form).

Theorem 1'. *$s_n \to s(J_q)$ and $s_n = \boldsymbol{O}(1)$ imply $s_n \to s(N, p^\alpha, q)$ for all $\alpha > 0$.*

Theorem 2'. *For $-1 < \alpha < \beta$:*

$s_n \to s(J_q)$ and $s_n = \boldsymbol{O}(1)\,(N, p^\alpha, q)$ imply $s_n \to s(N, p^\beta, q)$.

Theorem 3'. *Let*
$$\frac{p_n}{(p^\alpha * q)_n} \sim \mu_n(\alpha), \quad (n \to \infty)$$
with $(\mu_n(\alpha))$ *totally monotone for each* $\alpha \geq 0$ *. Then for bounded sequences* (s_n) *we have for* $0 < \alpha, \beta$
$$(N, p^\alpha, q) \approx (J_q, p^\beta),$$
where (J_q, p^β) *denotes the iteration product of the* (J_q) *and* (N, p^β, q) *methods.*

In this scale the (M_q)-methods are not directly included, but one can show (with Theorem 1 in [7]) that $(M_q) \subseteq (N, p^\alpha, q)$ for each $\alpha > 0$.

(ii) In [17] A.Tali discusses further convex families of general Nörlund transforms. These methods could likewise be used to construct scales of summability methods including power series methods.

(iii) Lemma 3 also allows a further application of the so-called modified reduction method introduced by M.R.Parameswaran in [14].

Proposition 3. *Let* (p_n) *satisfy* (1.4) *and* (V) *be a summability method such that*

(a) $na_n = \boldsymbol{O}(1)$ *is a tauberian condition for* (V) *to convergence for bounded sequences.*

(b) $(V) \subseteq (VN_{\alpha p})$ *for some* $\alpha \geq 1$ *.*

Then $(V) \subseteq (N_{\alpha p})$ *for bounded sequences.*

P r o o f: $s_n \to s(V)$ implies by (b) $s_n \to s(VN_{\alpha p})$. Under $s_n = \boldsymbol{O}(1)$ by Lemma 3 and (1.4) $(\sigma_n^{\alpha,p} - \sigma_{n-1}^{\alpha,p})n = \boldsymbol{O}(1)$. Hence by (a) $s_n \to s(N_{\alpha p})$.

(iv) For the case $p_n = \frac{1}{n+1}$, i.e the logarithmic method (L) , $s_n = \boldsymbol{O}(1)$ is not a tauberian condition of type $(J_p) \to (N, p, p)$ (compare Corollary 2). This can be seen as follows.

Set $L(n) := \sum_{k=0}^{n} \frac{1}{k+1}$ and

$$\tau_n := \frac{1}{L(n)} \sum_{k=0}^{n} \frac{1}{n-k+1} s_k \qquad \text{the Nörlund (harmonic) mean} \quad (N, \tfrac{1}{n+1}),$$

$$t_n := \frac{1}{L(n)} \sum_{k=0}^{n} \frac{1}{k+1} s_k \qquad \text{the weighted (logarithmic) mean} \quad (l),$$

$$\sigma_n := \frac{n+2}{2L(n)} \sum_{k=0}^{n} \frac{1}{n-k+1} \frac{s_k}{k+1} \qquad \text{the generalized Nörlund mean} \quad (N, \tfrac{1}{n+1}, \tfrac{1}{n+1}).$$

Straightforward computation show that $\tau_n + t_n = 2\sigma_n$. Hence whenever two of the above methods sum a sequence so does the third. It is also known [use e.g. 8, Theorem 23] that $(N, \frac{1}{n+1}) \subseteq (C_1)$.
From [8, §3.8 and Theorem 82] we know that there is a bounded sequence (A_n) which is summable (l) but not (C_1) . Now if boundedness were a tauberian condition, summability of (A_n) by (l) would imply summability $(N, \frac{1}{n+1}, \frac{1}{n+1})$ and via $(N, \frac{1}{n+1})$ summability (C_1) a contradiction.

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