Power series methods and almost sure convergence

by

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1 Introduction and main result

Let (Ω, Σ, P) be a probability space and suppose all random variables are defined on this space.
Let $X, X_0, X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) random variables and recall Kolmogorov's classical law of large numbers and its following extensions for further summability methods by respectively Lai [23] and by Chow [8] and Bingham and Tenenbaum [2]:

THEOREM L
The following are equivalent:

(i) $E(|X|) < \infty,\ E(X) = \mu$;

(ii) $$\frac{1}{n+1}\sum_{k=0}^{n} X_k \to \mu \quad (n \to \infty) \qquad \textit{a.s.; i.e. } X_n \to \mu(C_1) \ \textit{a.s.};$$

(iii) $$(1-t)\sum_{n=0}^{\infty} X_n t^n \to \mu \quad (t \to 1-) \qquad \textit{a.s.; i.e.} \quad X_n \to \mu(A) \ \textit{a.s.};$$

where (C_1) and (A) denote the classical Cesàro- and Abel-summability methods.

THEOREM C–B–T
The following are equivalent:

(i) $E(X) = \mu, \quad E(X^2) < \infty$;

(ii) $$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} X_k \to \mu \quad (n \to \infty) \qquad \textit{a.s.};$$

i.e., $X_n \to \mu(E_p)$ *a.s. for all (some)* $p \in (0,1)$;

(iii) $$e^{-t}\sum_{n=0}^{\infty} \frac{X_n}{n!} t^n \to \mu \quad (t \to \infty) \qquad \textit{a.s.; i.e.} \quad X_n \to \mu(B) \ \textit{a.s.}$$

(iv) $$\frac{1}{u\sqrt{n}} \sum_{k=n}^{n+u\sqrt{n}} X_k \to \mu \quad \forall\, u>0, \quad (n\to\infty) \quad a.s.;\ i.e.\ \ X_n \to \mu(R_{\sqrt{n}}) \quad a.s.,$$

with $(E_p),(B),(R_{\sqrt{n}})$ *the classical Euler- ,Borel- and moving–average means, see* [13] *or §3 below.*

Our aim is to extend and unify these theorems to the whole class of power series methods, where (A) and (B) are the best understood examples.
For this let us first define a power series method.

We assume that (p_n) is a sequence of reals with

$$(1.1)\quad \begin{cases} p_0>0, \quad p_n \geq 0, \quad n \in I\!N, \text{ such that the power series} \\ p(t)=\sum_{n=0}^{\infty} p_n t^n \quad \text{has radius of convergence } \ R>0. \end{cases}$$

For complex sequences (s_n) with increments $a_n = s_n - s_{n-1}, n\geq 0, s_{-1}=0$, we consider the power series method of summability (P), where we say

$$(1.2)\quad \begin{cases} s_n \to s(P), \quad \text{if } p_s(t)=\sum_{n=0}^{\infty} s_n p_n t^n \quad \text{converges for } |t|<R \\ \qquad\qquad\qquad \text{and if } \ \sigma_p(t)=\dfrac{p_s(t)}{p(t)} \to s \ \text{ as } \ t\to R-\,. \end{cases}$$

The power series methods contain the (J_p)- or Abel-type methods with $R=1$ and Abel's method, $p_n=1$, as a special case. The (B_p)- or Borel-type methods with $R=\infty$ and Borel's method, $p_n = 1/n!$, as a special case are similarly covered.

Since we can use $p_n R^n$ as weights in case $0<R<\infty$, we only have to deal with the two cases $R=1$ and $R=\infty$.
Furthermore we require the (P)-methods to be regular, i.e. $s_n \to s, (n\to\infty)$ implies $s_n \to s(P)$. By Theorem 5 in [13], we have regularity if and only if

$$(1.3)\quad \begin{cases} (A) \quad P_n=\sum_{k=0}^{n} p_k \to \infty, \ (n\to\infty), \text{ in case } R=1, \\ (B) \quad p(t) \text{ is not a polynomial, i.e.} \\ \qquad\ p_n \neq 0 \text{ for infinitely many } n \text{ in case } R=\infty. \end{cases}$$

A widely discussed field is the question of the reverse implication, namely the question, $s_n \to s(P)$ implies $s_n \to s \quad (n \to \infty)$. This implication is true only under additional assumptions (the so-called Tauberian conditions) and is called a Tauberian result. Tauberian results have a long history concerning Abel's and Borel's method, see [13,24,27] and have recently found new attention for special classes, see [10,15,16]. In a series of papers Kratz,Stadtmüller and other authors [7,19,20,21,22] developed a general theory of Tauberian results for regular power series methods, especially linking Tauberian results for Abel- and Boreltype methods. To formulate our main result, we need some notation and results of Kratz,Stadtmüller (for details see §2 below).

First we introduce the sequence of maximal weights" (Δ_n) defined by

$$\Delta_n = \inf_{0<t<R} p(t)t^{-n}. \tag{1.4}$$

This sequence is related to the limitation order of the power series method, see [7], and plays a decisive role in formulating Tauberian conditions.
In order to prevent the weights p_n from being too irregular, we use the following condition

$$\Delta_n \sim l(n)\, p_n \quad (n \to \infty), \tag{1.5}$$

with $l(t)$ a continuous, strictly increasing function.
In case $l(t) = \boldsymbol{o}\,(t)$, we assume furthermore $l(.)$ as self-neglecting, see [6] §2.11 for the definition, and

$$p_n \sim \exp\{g(n)\}, \tag{1.5'}$$

with $g(.)$ a logarithmico- exponential function, satisfying certain growtth-conditions, see [10,21,22] and the Remark below.

We are now in the position to state our main result

THEOREM

Let (P) be a regular power series method satisfying (1.5). *Then the following statements are equivalent:*

(i) $E(l^{-1}(|X|)) < \infty$ *;*

(ii) $(X_n - C_n) \to 0\ (S)$ *a.s., where* (S) *is some weaker summability method depending on* (P) *;*

(iii) $(X_n - C_n) \to 0 \quad (P)$ *a.s.;*
where $C_n = E\left(X I_{[|X| \le l(n)]}\right)$ *resp., if* $E(|X|) < \infty,\ C_n = E(X)$.

Remark 1

(i) Since $\Delta_n \sim en$ resp. $\Delta_n \sim \left(\frac{e}{n}\right)^n$ in the Abel resp. Borel case Theorem C and L–B–T are included.

(ii) Observe that (1.5) together with the regularity condition implies $p_n = \boldsymbol{o}\,(\Delta_n)$.

(iii) In case $\alpha < 1$ we cover the (J_p)–case with rapidly increasing weights, i.e. $p_n \sim \exp\{g(n)\}$, with $tg'(t) \to \infty \ \ (t \to \infty), g''(t) < 0$ and the (B_p)–case, i.e. $p_n \sim \exp\{-g(n)\}$, with $g'(t) \to \infty$ and $g''(t) \to \infty \ (t \to \infty)$, see the examples in §2.

(iv) We show in the sequel that our moment-condition covers the whole range of moment-conditions from slightly weaker than first moment to moments of arbitrarly high order.

We start in §2 to give the main results concerning power series methods developed by Kratz,Stadtmüller.
In §3 we discuss the connection between power series methods and some classes of weaker summability methods, e.g. the methods of weighted means, the moving average methods and the Riesz-methods. We prove especially a new result of abelian type from moving averages to power series methods, which applies to a large class of such methods, including and generalizing results in [26].
§4 is devoted to the probabilistic parts of our arguments. We generalize a strong law for moving averages, which was first given by Chow in [8].
Finally in §5, we prove our Theorem.

2 Powers series methods and dominated variation

We give some informations on (Δ_n) useful in our context. For a detailed discussion and proofs see [7,19,20,22].

LEMMA 1
We have

(2.1) $\Delta_n = p(t_n)t_n^{-n}$ *for* $n = 0, 1, 2, \ldots,$ *such that*

(i) (t_n) *is non-decreasing and* $\lim_{n\to\infty} t_n = R$ *;*

(ii) $t_m^{n-m} \le \Delta_m/\Delta_n \le t_n^{n-m}$ *for* $n, m \in I\!N$ *;*

(iii) $p_n/\Delta_n \le 1$ *and* $\displaystyle\sum_{n=0}^{\infty} \frac{p_n}{\Delta_n} = \infty$.

Remark 2

(i) Observe that (iii) is a generalization of the Abel-Dini-theorem.

(ii) In case $p_n \sim \exp(g(n))$, with some logaritmico-exponential function $g(.)$, we have under the regularity conditions on the order of growth of $g(.)$, see Remark 1:

$$l(n) \sim c \left\{ \frac{|g''(n)|^{-\frac{1}{2}}}{2\pi} \right\},$$

with c some constant.

LEMMA 2
Assume (1.1)*. Then the following assertions are equivalent:*

(i) $P(x) = \displaystyle\sum_{0\le k\le x} p_k$ *is dominatedly varying,*

i.e $\displaystyle\limsup_{\lambda\to\infty} \frac{P(\lambda x)}{P(\lambda)} < \infty$ *for all (some)* $x > 1$ *;*

(ii) $\Delta_n = \boldsymbol{O}(P_n)$ *;*

(iii) $\displaystyle\frac{np_n}{\Delta_n} = \boldsymbol{O}(1)$.

Proof:
The following equivalences were obtained in [20, Corollary 3].

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \frac{p(1-c/n)}{\Delta_n} = \boldsymbol{O}(1) \text{ for all (some) } c>0, \quad \Leftrightarrow \frac{np_n}{P_n} = \boldsymbol{O}(1)\,.$$

That (iii) implies (i) follows by:

$$\begin{aligned} p(1-c/n) \quad &\le \Delta_n + \frac{c}{2n}\sum_{k=n}^{\infty}\Delta_k(1-c/n)^k \\ &\le \Delta_n + \frac{cp(1-c/n)}{2n}\sum_{k=n}^{\infty}(1-c/n)^k \le \Delta_n + \frac{1}{2}p(1-c/n). \end{aligned}$$

The implication from (i) to (iii) is a trivial application of the basic inequality $P_n \le \Delta_n$.
■

Examples:
Throughout we will illustrate our results on some special methods, which we introduce here. For properties and further references on the methods see [13,24,26].

Abel–type

(i) Generalized Logarithmic Methods (L_α) :

$p_n = \dfrac{\alpha(\log^{\alpha-1}(n+1))}{n+1}$, $\alpha > 0$; $\Delta_n \sim c\log n$ with $\alpha = 1$ being the classical logarithmic method (L) .

(ii) Generalized Abel Methods (A_α) :

$p_n = \dbinom{n+\alpha}{n}$; $\Delta_n \sim cn$ with $\alpha = 0$ being the classical Abel method (A) .

(iii) (J_p) Methods with regularly varying weights (A_{R_α}) :
$p_n = n^\alpha L(n)$, $\alpha > -1$, $L(.)$ slowly varying, see [25], $\Delta_n \sim cn$ with $\alpha = 0$, $L(t) \equiv 1$ being the classical Abel method (A) .

(iv) Methods with rapidly increasing weights (A_p) :
$p_n = \exp\{n^\alpha\}$, $0 < \alpha < 1$, $\Delta_n \sim cn^{1-\frac{\alpha}{2}}p_n$.

Borel–type

(v) Generalized Borel-methods $(B(\alpha,\beta))$:

$p_n = \dfrac{\alpha}{\Gamma(\alpha n+\beta)}$, t replaced by t^α, $\alpha,\beta>0$, $\Delta_n \sim c\sqrt{n}p_n$ with $\alpha=\beta=1$ the classical Borel-method (B).

(vi) (B_p)–Methods:

$p_n = \exp\{-n^\beta\}$, $1<\beta<2$, $\Delta_n \sim cn^{1-\frac{\beta}{2}}p_n$.

3 Definition and properties of auxiliary summability methods

In addition to the power series methods, we shall principally be concerned with three classical important classes of summability methods.

For a complex sequence (s_n), $s_n = \sum_{k=0}^{n} a_k$ recall the weighted mean average (M_p) based on a sequence (p_n) of nonnegative reals, which satisfies $p_0 > 0$ and (1.3A). Then we say $s_n \to s(M_p)$ if

$$\frac{1}{P_n}\sum_{k=0}^{n} s_n p_n \to s, \quad (n \to \infty). \tag{3.1}$$

Secondly recall the Riesz (typical) means $(R, \lambda_n, 1)$, of order 1 based on a real sequence $0 \leq \lambda_0 < \lambda_1 < \ldots$. We say $s_n \to s(R, \lambda_n, 1)$ if

$$\frac{1}{t}\int_0^t \left\{ \sum_{\lambda_k \leq x} a_k \right\} dx \to s, \; (t \to \infty). \tag{3.2}$$

Finally, we need the methods of moving averages (R_ϕ), where $\phi(.)$ is a self-neglecting function.
We say $s_n \to s(R_\phi)$ if

$$\frac{1}{u\phi(n)} \sum_{n \leq k < n+u\phi(n)} s_k \to s, \quad \forall u > 0, \quad (n \to \infty). \tag{3.3}$$

The relations of this methods to each other are (see e.g. [23] resp. [4])

LEMMA 3

(i) $(R, P_n, 1) \approx (M_p)$.

(ii) $(R, \exp\{\Phi(n)\}, 1) \approx R_\phi$, *with* $\Phi(x) := \int_1^x dt/\phi(t) \quad (x \geq 1)$.

Notation:
For two summability methods $(A), (B)$: $(A) \approx (B)$ means
$s_n \to s(A) \Rightarrow s_n \to s(B)$ and vice versa.

In our context we are interested in Riesz-means with the sequences (λ_n) of special type, namely

$$\lambda_n = \exp\left\{ \int_1^n \frac{1}{l(t)} dt \right\}$$

with $l(t)$ satisfying the conditions in (1.5). The second theorem of consistency, see [14, Theorem 19], easily implies the following equivalences:

LEMMA 4

(i) $(R, \log(n+1), 1) \approx (R, \exp\left\{\int\limits_1^{n+1} \frac{1}{x \log x} dx\right\}, 1)$,

(ii) $(R, n, 1) \approx (R, \exp\left\{\int\limits_1^{n} \frac{1}{x} dx\right\}, 1)$,

Remark 3

Combining the above Lemmas, we get connections between (M_p)-means and Riesz-means with special (λ_n) sequences.

We now prove abelian results relating the above methods with (P)-methods.

PROPOSITION 1

(i) *Let* (p_n) *be a sequence of reals satisfying* (1.1) *and* (1.3A)*, then we have for the corresponding power-series method* (P) *and the* (M_p) *mean* $s_n \to s(M_p)$ *implies* $s_n \to s(P)$.

(ii) *Let* (P) *be a power series method satisfying* (1.5) *with* $l(t) = \boldsymbol{o}(t)$, *then*

$$(3.4) \quad \frac{1}{xl(n)} \sum_{n \le k \le n + xl(n)} s_k \to s, \quad (n \to \infty), \forall x > 0 \quad \textit{implies } s_n \to s(P).$$

Proof:

(i) see Ishiguro [15].

(ii) Tenenbaum [26, Lemma 2] gives the above result for the Borel-method, e.g. $l(t) = \sqrt{t}$. His proof uses heavily the special asymptotic properties of the weights $p_n = 1/n!$. Since our restriction (1.5) together with (1.5′) and $l(t) = \boldsymbol{o}(t)$ basically imply the same asymptotic properties for the weights, the main steps of his proof extend and we give only a short outline.

Because of the regularity of the methods, we can assume $s = 0$.

As a first step we use Theorem 1 in Bingham/Goldie [3] to get (3.4) uniformly on compact x-sets in $(0, \infty)$.

Then we define a nondecreasing, differentiable function $h : [0, \infty) \to \mathbb{R}$ such that $h(n) = \log t_n$, $n \in \mathbb{N}$, with t_n as in Lemma 1, hence $h(t) \to \log R$, $(t \to \infty)$. With h we carry out a parameter transformation and set

$$c_n(t) = \frac{p_n \exp\{nh(t)\}}{p(\exp\{h(t)\})}, \quad n = 0, 1, 2 \ldots; \ t > 0.$$

This transformation was introduced in the context of Tauberian theorems, see [18].

The properties of Δ_n for power series methods with weights according to (1.5') imply:

$$(h'(t))^{-\frac{1}{2}} \sim l(t) \;\; (t \to \infty).$$

Now we can use the basic inequality in Lemma 1(ii) and the above asymptotic relation for $l(.)$ to obtain through an application of the mean-value theorem that

$$c_n(t) = \boldsymbol{O}\left(\tfrac{1}{l(t)}\right) \text{ for } |n-t| = \boldsymbol{O}\left(l(t)\right). \tag{3.5}$$

Introducing a further function $\epsilon(t)$, see [7,21,22], with

$$\frac{\epsilon(t)}{l(t)} \to \infty, \quad \frac{\epsilon(t)}{t} \to 0, \quad (t \to \infty),$$

we can split $\sum\limits_{n=0}^{\infty} s_n c_n(t)$ in five parts, namely (writing m for $[t]$):

$$\begin{aligned}\sum_{n=0}^{\infty} s_n c_n(t) &= \left\{ \sum_{n=0}^{m-\epsilon(m)-1} + \sum_{n=m-\epsilon(m)}^{m-l(m)} + \sum_{n=m-l(m)+1}^{m+l(m)-1} + \sum_{n=m+l(m)}^{m+\epsilon(m)} + \sum_{n=m+\epsilon(m)+1}^{\infty} \right\} s_n c_n(t) \\ &= \textstyle\sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5 .\end{aligned}$$

Using Abel's partial summation, the uniformity in (3.4) and the basic growth condition in (3.5), we obtain by the monotonicity of $c_n(t)$ in n for fixed t :

$$\sup_{|n-m| = \boldsymbol{O}(l(m))} |c_{n+1}(t) - c_n(t)| = \boldsymbol{O}\left(\frac{1}{l(m)}\right) \boldsymbol{o}\left(l(m)h'(m)\right) = \boldsymbol{O}\left(\frac{1}{(l(m))^2}\right).$$

Now we obtain by Abel's partial summation $\sum_3 = \boldsymbol{o}(1)$. That $\sum_1 + \sum_5 = \boldsymbol{o}(1)$ follows again via Abel's partial summation and the fact that $\sum\limits_{k=0}^{n} s_k = \boldsymbol{o}(n)$. Use $l(n) = \boldsymbol{o}(n)$ and (3.4) to obtain this.

To get $\sum_2 + \sum_4 = \boldsymbol{o}(1)$, we use the basic property $\left|1 - \dfrac{l(n)}{l(m)}\right| \leq 3\dfrac{\epsilon(m)}{m}$ for $|m-n| \leq \epsilon(m)$, compare [7,18,21,22,23].

■

Remark 4

Combining Propositon 1 (ii) and Remark 3, we obtain implications from (M_q) -methods to (P) -methods.

4 Strong Laws for the auxiliary summability methods

In this paragraph we formulate and extend strong law type results for the moving average methods and the (M_p)-means. These results are the basic tool for the proof of the first implication in our Theorem. We first need some notation:

$$(4.1)\quad \begin{cases} \bar{S}_{n,l(n)} = \max\limits_{0\le j\le l(n)} \sum\limits_{k=n}^{n+j} X_k \text{ and} \\ S^*_{n,l(n)} = \max\limits_{0\le j\le l(n)} \left| \sum\limits_{k=n}^{n+j} X_k \right| \end{cases}$$

Then we have

PROPOSITION 2

Let $l(t)$ be a length-function as in (1.5)*, with $l(t) = \boldsymbol{o}(t)$.*
Assume $E(l^{-1}(|X|)) < \infty$ and $E(X) = 0$. Then we have

(i) $\limsup\limits_{n\to\infty} \frac{1}{l(n)} \bar{S}_{n,l(n)} = 0 \quad a.s.;$

(ii) $\limsup\limits_{n\to\infty} \frac{1}{l(n)} S^*_{n,l(n)} = 0 \quad a.s.$

Proof:

(i) We can follow the lines of the proof of Theorem 1 in [8]. There the case $l(t) = t^\alpha, 0 < \alpha < 1$ is treated. Since we have $l(t)$ is self-neglecting, we only have to use the special properties of self-neglecting functions to adjust the arguments, see [4] resp.[17] for the details.

(ii) Use the sequences (X_n) and $(-X_n)$, then (i) implies (ii).

For convenience we add a result basically proved in [9], §5.2, Theorem 3.

LEMMA 5

Let (p_n) a sequence of reals satisfying (1.1) *and* (1.3A) *and (p_{n_ν}) the nonzero subsequence. Assume furthermore that the sequence $0 < \dfrac{P_{n_\nu}}{p_{n_\nu}}$ is strictly increasing and that*

$$(4.2)\qquad \left(\frac{P_{n_\nu}}{p_{n_\nu}}\right)^2 \sum_{k=\nu}^{\infty} \left(\frac{p_{n_k}}{P_{n_k}}\right)^2 = \boldsymbol{O}(\nu).$$

Then we have

$$\sum_{n=0}^{\infty} P(p_n|X| > P_n) < \infty \;\Rightarrow\; \frac{1}{P_n}\sum_{k=0}^{n} p_k(X_k - E(XI_{[p_k|X_k|\le P_k]})) \to 0 \ \text{a.s.}, \ \textit{that is}$$

$$(X_n - E(XI_{[p_n|X_n|\le P_n]})) \to 0(M_p),$$

with I *the indicator function.*

5 Proof of the main theorem

Before we prove the Theorem we need as a final preparation a Proposition, useful in its own right, which investigates the relation between (P)- convergence and moment conditions.

PROPOSITION 3

Let (P) be a regular power series method with a sequence of weights (p_n), satisfying (1.1) and (1.5). Let as usual (X_n) be a sequence of i.i.d. random variables and (C_n) a sequence of real numbers.

Then $(X_n - C_n) \to 0(P)$ implies $\sum_{n=0}^{\infty} P(p_n|X_n| > \Delta_n) < \infty$.

Proof:

Let (X'_n) and (X_n) be i.i.d. random variables and $X_n^s = X_n - X'_n$, then we have $X_n^s \to 0(P)$. With $\Delta_m = p(t_m)t_m^{-m}$, see (2.1), we define

$$Y_m = \sum_{n=0}^{m} \frac{p_n t_m^n}{p(t_m)} X_n^s \quad \text{and} \quad Z_m = \sum_{n=m+1}^{\infty} \frac{p_n t_m^n}{p(t_m)} X_n^s$$

for $m = 0, 1, \ldots$. It follows that $(Y_1, Y_2, \ldots Y_m)$ and Z_m are independent for all m, and Y_m and Z_m are symmetrical. By Lévy's second inequality and with $Y_m + Z_m \overset{\text{P}}{\to} 0$, we get $Z_m \overset{\text{P}}{\to} 0$. We now use Lemma 4, p.70 in [9], to obtain $Y_m \overset{\text{a.s.}}{\to} 0$. Repeating the argument for Y_{m-1} and $Z_m = Y_m - Y_{m-1}$ leads to

$$\frac{p_m t_m^m}{p(t_m)} X_m^s = \frac{p_m}{\Delta_m} X_m^s \overset{\text{a.s.}}{\to} 0.$$

An easy application of the Borel–Cantelli–Lemma and the symmetrisation inequality [11, p.102] results in

$$\frac{p_m}{\Delta_m}(X_m - m(X)) \overset{\text{a.s.}}{\to} 0$$

with $m(X)$ denoting the median of X. Since $p_m = \boldsymbol{o}(\Delta_m)$ is a consequence of (1.5) the proof is completed.

■

Proof of the Theorem:
We distinguish two cases.
(a) $n = \boldsymbol{O}\,(l(n))$
Here we use as weaker summability method the corresponding (M_p)-means. Since (a) implies $\dfrac{np_n}{\Delta_n} = \boldsymbol{O}\,(1)$ the P_n are dominatedly varying and we have $P_n \leq \Delta_n = \boldsymbol{O}\,(P_n)$ by Lemma 2. Therefore

$$E(l^{-1}(|X|)) < \infty \Leftrightarrow \sum_{n=0}^{\infty} P(p_n|X_n| > \Delta_n) < \infty \Leftrightarrow \sum_{n=0}^{\infty} P(p_n|X_n| > P_n) < \infty,$$

and we can use Lemma 5 to obtain the implication from (i) to (ii), i.e

$$X_n - C_n \to 0(M_p)\ a.s.$$

with $C_n = E(XI_{[p_n|X_n|<\Delta_n]})$ or if $E(|X|) < \infty$, $C_n = E(X)$.
Now Ishiguro's Theorem, Proposition 1(i), implies

$$X_n - C_n \to 0(P)\ a.s.$$

which is (iii). The concluding implication from (iii) back to (i) follows by Proposition 3.

(b) $l(n) = \boldsymbol{o}\,(n)$.
Now we use as weaker summability method the moving averages as defined in (3.4). With Proposition 2 we get the implication from $E(l^{-1}(|X|)) < \infty$ to convergence in (3.4), while Proposition 1 gives the convergence in the (P)-sense, i.e. $X_n - E(X_n) \to 0(P)$ a.s. As above Proposition 3 concludes the proof.
■

Restricting the class of power series methods we get the following

Corollary:
Assume

$$(P) \in \{(L_\alpha), (A_\alpha), (A_{R_\alpha}), (A_p), (B(\alpha,\beta)), (B_p)\}.$$

Then the following are equivalent
(i) $E(l^{-1}(|X|)) < \infty$

(ii) $X_n - C_n \to 0 \quad \left(R, \exp\left\{\int_1^n \frac{1}{l(t)}dt\right\}, 1\right)\ a.s.$

(iii) $X_n - C_n \to 0(P)\ a.s$ *with* $C_n = E(XI_{[p_n|X_n|<\Delta_n]})$
or if $E(|X|) < \infty, C_n = E(X)$.

Proof:
We only have to show the equivalence of the summability methods used for (ii). We consider three cases.

(a) $(P) \in (L_\alpha)$:
We have $n = \boldsymbol{O}(l(n)) = \boldsymbol{O}(n \log n)$. In the proof of Theorem 3 we used the corresponding (M_p)-means. Since all logarithmic-type (M_p)- means are equivalent to the logarithmic mean with $p_n = 1/n$, use [13] Theorem 14, we obtain our result by Lemma 3(i) and Lemma 4(i).

(b) $(P) \in \{(A_\alpha), (A_{R_\alpha})\}$:
Since now $n = \boldsymbol{O}(l(n)) = \boldsymbol{O}(n)$ again the corresponding (M_p)-means have been used. Here all (M_p)- means are equivalent to the Cesàro mean with $p_n = 1$, again by Theorem 14 in [13]. Another application of Lemma 3(i) and Lemma 4(ii) gives the remaining equivalences.

(c) $(P) \in \{(A_p), (B(\alpha,\beta)), (B_p)\}$:
Since now $l(n) = \boldsymbol{o}(n)$ the moving averages have been used in Theorem 3. But these are equivalent to the given Riesz-means by Lemma 3(ii).

■

Remark 5:

(i) In the above Corollary the (M_p)-means can also be used as auxiliary methods (see Remark 3).

(ii) We give a survey of our results in terms of moment conditions, the given moment condition being equivalent to convergence in the sense of the corresponding summability method.

(L_α) $E\left(\dfrac{|X|}{\log^+|X|}\right) < \infty$. Here we have to adjust the X_n by subtracting $C_n = E\left(X I_{[|X| < \frac{n}{\alpha}\log n]}\right)$.

(A_α), (A_{R_α}) $E(|X|) < \infty$;

(A_p) $E\left(|X|^{\frac{2}{2-\alpha}}\right) < \infty$, $p_n = \exp\{n^\alpha\}, 0 < \alpha < 1$;

$(B(\alpha,\beta))$ $E\left(|X|^2\right) < \infty$;

(B_p) $E\left(|X|^{\frac{2}{2-\beta}}\right) < \infty$, $p_n = \exp\{-n^\beta\}, 1 < \beta < 2$.

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