Tauberian theorems for general power series methods

by

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1 Introduction and main result

Let us assume throughout that (p_n) denotes a sequence of reals which satisfies

$$(1.1)\begin{cases} p_0 > 0,\;\; p_n \geq 0, \quad n = 1,2,\ldots \quad \text{,such that the power series} \\ p(t) = \sum_{n=0}^{\infty} p_n t^n \quad \text{has radius of convergence } R \in (0,\infty]. \end{cases}$$

For real sequences (s_n) with increments $a_n = s_n - s_{n-1}, n \geq 0$, $(s_{-1} = 0)$, we consider the **power-series method of summability** (P), where we say

$$(1.2)\qquad \begin{cases} s_n \to s(P) \quad \text{if } p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \quad \text{converges for } |t| < R \\ \text{and if} \quad \sigma_p(t) = \dfrac{p_s(t)}{p(t)} \to s \quad \text{as} \quad t \to R- . \end{cases}$$

The power series methods (P) contain the so–called (J_p)–methods ($R = 1$) and the Borel–type methods (B_p) ($R = \infty$). We consider only regular (P)–methods, i.e. $s_n \to s$ implies $s_n \to s(P)$. By Theorem 5 in [5], p.49, we have regularity iff

$$(1.3)\quad \begin{cases} \sum_{k=0}^{n} p_k R^k \to \infty, \quad (n \to \infty) \quad \text{in case} \quad R < \infty, \\ (\text{ in particular } \; P_n = \sum_{k=0}^{n} p_k \to \infty, \quad (n \to \infty) \;\text{ if } \; R = 1,) \\ p(t) \text{ is not a polynomial, i.e.} \quad p_n \neq 0 \quad \text{infinitely often in case } \; R = \infty. \end{cases}$$

Here we are interested in the converse conclusion, namely $s_n \to s(P)$ implies $s_n \to s$, which can only be valid if further conditions, so called Tauberian conditions are satisfied by (s_n). These so-called **Tauberian–theorems** for power series methods have a long history, see e.g the books [5,14,23], and found new attention recently in the papers [6,18,19,20] and [8,9,10,11,12]. The latter papers contain certain $\boldsymbol{o}$–Tauberian theorems for all power series methods in question and $\boldsymbol{O}$–Tauberian theorems, if the weight sequence (p_n) can be interpolated by a logarithmico–exponential function $g(.)$, see e.g. [4], i.e.

$$p_n \sim \exp\{g(n)\}. \tag{1.4}$$

In order to state these results we need the following quantities:

$$\Delta_n = \inf_{0<t<R} p(t)t^{-n}, \quad n = 0, 1, 2, \dots, \tag{1.5}$$

which by the results in [2,12] can be written as

$$\Delta_n = p(t_n)t_n^{-n} \quad \text{for} \quad n = 0, 1, 2, \dots \quad \text{with some sequence} \quad 0 < (t_n) \nearrow R. \tag{1.6}$$

Then an $\boldsymbol{o}$–Tauberian condition, valid for all methods satisfying (1.1) and (1.3) is given, see [9,10,12], by

$$\lim_{n\to\infty} \max_{n\le m\le n+l(n)} |s_{m+1} - s_n| = 0 \tag{1.7}$$

where $(l(n))$ is a sequence of integers s.t.

$$\sum_{k=n}^{n+l(n)} \frac{\Delta_k}{p_k} \ge 1, \quad n \in I\!N. \tag{1.8}$$

Under the additional asumption (1.4) we can weaken this Tauberian condition to the following $\boldsymbol{O}$–condition, see [11,12]:

$$\omega(\lambda) := \limsup_{n\to\infty} \max_{n\le m\le n+l(n,\lambda)} |s_{m+1} - s_n| \to 0, \quad (\lambda \to 1+), \tag{1.9}$$

where $(l(n,\lambda))$ now is for each $\lambda > 1$ a sequence of integers s.t.

$$\sum_{k=n}^{n+l(n,\lambda)} \frac{\Delta_k}{p_k} \ge \lambda - 1, \quad n \in I\!N. \tag{1.8$'$}$$

If we denote in (1.8) or (1.8$'$) the shortest possible by sequences $(l^*(n)), (l^*(n,\lambda))$ resp. (for a discussion see [10,11]), then we can equivalently choose the sequences $(cl^*(n)), (cl^*(n,\lambda))$ resp. for any $c > 0$, provided (1.4) holds. Hence in case of:

(i) the **Abel method** with $p_n = 1, R = 1$:
$\Delta_n \sim en$ and $l(n,\lambda)$ can be choosen as $(\lambda - 1)n$, see e.g. [5,16].

More general, if $p_n = R(n)$ with a regularly varying function R(.), see [1,17], with index $\alpha > -1$, then again $\Delta_n \sim c(\alpha)n$ and $l(n,\lambda) = (\lambda - 1)n$ is a possible choice, see e.g. [6,10,11].

(ii) the **Borel method** with $p_n = \dfrac{1}{n!}, R = \infty$:
$\Delta_n = \left(\dfrac{e}{n}\right)^n$ and $l(n,\lambda)$ can be choosen as $(\lambda - 1)\sqrt{n}$, see e.g. [5].

In all this special cases a one–sided version of (1.9) is a sufficient Tauberian condition (TC) as well, see [6,16,18,21,22], i.e.

$$(1.9') \qquad \omega(\lambda) := \liminf_{n\to\infty} \min_{n\le m\le n+l(n,\lambda)} (s_{m+1} - s_n) \ \text{ satisfies } \ \lim_{\lambda\to 1+} \omega(\lambda) \ge 0\,,$$

where $(l(n,\lambda))$ is defined as in $(1.8')$.

The proofs of all one–sided results contain the following steps:

(1) the Vijayaraghavan–type conclusion:
$s_n \to s(P)$ plus (TC) implies $s_n = \boldsymbol{O}(1)$;

(2) the weaker summability method step:
$s_n \to s(P)$ plus $s_n = \boldsymbol{O}(1)$ implies $s_n \to s(S)$,
where (S) stands for a (or a class of) weaker summability–method(s);

(3) the direct argument:
$s_n \to s(S)$ plus (TC) implies $s_n \to s$.

The main result in this paper is a Vijayaraghavan–type theorem giving the conclusion (1) for all power series methods in question. The proof uses the general properties of the sequences (Δ_n) and (t_n) and avoids in contrast to the literature special asymptotic considerations.

THEOREM 1
Let be given a regular power–series method (P), obeying (1.1) and (1.3) with a sequence of weights (p_n). Suppose further, that (s_n) is a sequence of reals, satisfying

$$s_n = \boldsymbol{O}(1)\,(P), \ \text{ i.e. } \frac{p_s(t)}{p(t)} = \boldsymbol{O}(1)\,, \quad (t\to R-)$$

and

$$(1.9'') \qquad \omega(\lambda) = \liminf_{n\to\infty} \min_{n\le m\le n+l_n} (s_{m+1} - s_n) > -\infty,$$

where (l_n) is a sequence of nonnegative integers with

$$(1.8'') \qquad \liminf_{n\to\infty} \left(\sum_{k=n}^{n+l_n} \frac{p_k}{\Delta_k} \right) > 0.$$

Then we have $s_n = \boldsymbol{O}(1)$.

Remark 1:

(i) In $(1.9'')$ we may use any of the sequences $(l(n)), (l(n,\lambda)), (l^*(n)), (l^*(n,\lambda))$ for any $\lambda > 1$.

(ii) The following cases have been discussed in the literature: (In what follows $c > 0$ stands for some constant.)

(a) the Logarithmic method, [13]:
$p_n = \dfrac{1}{n}, R = 1, l(n) = c\, n \log n$.

(b) the Abel method, [16,21], or more generally (J_p)–methods with polynomial weights, [6,18]:
$p_n = 1$, or $p_n = R(n)$, $R(.)$ some regularly varying function with Index $\alpha > -1$. Then we have $R = 1$ and $l(n) = cn$.

(c) (J_p)–methods with exponential increasing weights, [7]:
$p_n = e^{n^\alpha},\ 0 < \alpha < 1,\ R = 1,\ l(n) = cn^{1-\frac{\alpha}{2}}$.

(d) the Borel method, [22]:
$p_n = \dfrac{1}{n!},\ R = \infty,\ l(n) = c\sqrt{n}$.

(e) the Generalized Borel–methods, [3]:
$p_n = \dfrac{\alpha}{\Gamma(\alpha n + \beta)}$ (with t replaced by t^α), $\alpha, \beta > 0,\ l(n) = c\sqrt{n}$.

(f) (B_p)–methods, [7]:
$p_n = e^{-n^\beta},\ 1 < \beta < 2,\ R = \infty,\ l(n) = cn^{1-\frac{\beta}{2}}$.

In section 2 we shall prove Theorem 1, whereas in section 3 we give a Tauberian theorem under the one–sided condition $(1.9')$ for all power series methods, provided (1.4) is valid.

2 Proof of Theorem 1

The main tools for the proof are a Vijayaraghavan type theorem and the results about the quantities Δ_n (see (1.5)) stated in [2,9,12]. In what follows we consider the method of summation (χ) defined by

$$\chi(t) = \sum_{n=0}^{\infty} c_n(t) s_n \to s \tag{2.1}$$

for $t \to \infty$ and (s_n) a sequence of reals. We suppose that

$$c_n(t) \geq 0 \quad (t > 0, n \geq 0), \quad c_n(t) \to 0, \quad \sum_{n=0}^{\infty} c_n(t) \to 1, \quad (t \to \infty) \tag{2.2}$$

so that (χ) is regular.
Then we have the following Vijayaraghavan type theorem.

LEMMA 1
Consider the summability method (2.1) satisfying (2.2) and a real sequence (s_n). If there exists a positive, strictly increasing function $\Phi(.)$ on $(0, \infty)$ s.t

$$\Phi(t+1) - \Phi(t) \leq 2 \quad \text{and} \quad \Phi(t) \to \infty, \quad (t \to \infty) \tag{2.3}$$

and

$$\sum_{n=0}^{M} c_n(t) \to 0\,, \text{ as } \Phi(t) - \Phi(M) \to \infty, \quad t > M \to \infty; \tag{2.4}$$

$$\sum_{n=N}^{\infty} c_n(t)(\Phi(n) - \Phi(N)) \to 0\,, \text{ as } \Phi(N) - \Phi(t) \to \infty, \quad N > t \to \infty; \tag{2.5}$$

(2.6) and if there exist positive constants a, b s.t. for $u > t > 0$

$$s(u) - s(t) > -a\{\Phi(u) - \Phi(t)\} - b,$$

where $s(t) = s_n$ for $n \leq t < n+1$,

then $s_n = \boldsymbol{O}(1)\,(\chi)$, implies $s_n = \boldsymbol{O}(1)$.

Proof:
One has to modify slightly the proof of Theorem 238 in [5] or of Lemma 1 in [15], but we demand only the boundedness of $\Phi(t+1) - \Phi(t)$ rather then the convergence of this difference to zero.

Proof of Theorem 1:
We want to use Lemma 1, therefore it is useful to rescale the (P)–methods. Starting with the sequences (t_n) as in (1.6), we define a nondecreasing function $h:[0,\infty)\to I\!R$ s.t. $h(n)=\log t_n$, $n\in I\!N$, hence $\lim_{t\to\infty} h(t)=\log R$. Then we put for a given power series method (P)

$$c_n(t)=\frac{p_n\exp\{nh(t)\}}{p(e^{h(t)})}\ ,\quad t>0,\quad n=0,1,2,\ldots$$

By (1.1) and (1.3) $c_n(t)$ satisfies (2.2). Next we define a function $\Phi(.)$, beginning with $f(t)=\sum_{k=0}^{[t]}\frac{p_k}{\Delta_k}$, where as usual $[t]=n$, if $n\le t<n+1$ for some $n\in I\!N_0$.
Then we replace the step function $f(.)$ by a strictly increasing function $\Phi:(0,\infty)\to(0,\infty)$, which interpolates $f(.)$ at all points $t=n$, where $p_n>0$. $\Phi(.)$ satisfies (2.3) as desired, since

$$\frac{p_k}{\Delta_k}\le 1$$

and

$$\sum_{k=0}^{\infty}\frac{p_k}{\Delta_k}=\infty$$

by Lemma 1 in [9,12]. With these quantities we check the conditions (2.4) – (2.6).

ad (2.4) :

Observe that by (1.6), see also [9,12]

$$p\left(e^{h(n)}\right)\ e^{-nh(n)}=\Delta_n\le\Delta_k e^{(k-n)h(k)},$$

i.e. $\Delta_k\ge\Delta_n e^{-(k-n)h(k)}$ for all $k,n\in I\!N$.
Hence for $t>M>0$, $M\in I\!N$, we obtain by the monotonicity of $h(.)$:

$$\begin{aligned}\Phi(t)-\Phi(M)\ &\le\ \sum_{n=M+1}^{[t]+1}\frac{p_n}{\Delta_n}\le\frac{1}{\Delta_M}\sum_{n=M+1}^{[t]}p_n e^{(n-M)h(n)}+1\\ &\le\ \frac{1}{\Delta_M}\sum_{n=M+1}^{[t]}p_n e^{(n-M)h(t)}+1=\frac{e^{-Mh(t)}}{\Delta_M}\sum_{n=M+1}^{[t]}p_n e^{nh(t)}+1.\end{aligned}$$

So from (2.4) we can assume that the first term tends to infinity as $t>M\to\infty$. This implies (use again the monotonicity of $h(.)$ and (1.6)):

$$
\begin{aligned}
\sum_{n=0}^{M} c_n(t) &= \sum_{n=0}^{M} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} = \sum_{n=0}^{M} \frac{p_n e^{nh(M)+n(h(t)-h(M))}}{p\left(e^{h(t)}\right)} \\
&\leq \frac{e^{M(h(t)-h(M))}}{p\left(e^{h(t)}\right)} \sum_{n=0}^{M} p_n e^{nh(M)} \\
&\leq \frac{e^{-Mh(M)} \sum_{n=0}^{\infty} p_n e^{nh(M)}}{p\left(e^{h(t)}\right) e^{-Mh(t)}} \\
&\leq \frac{\Delta_M}{e^{-Mh(t)} \sum_{n=M+1}^{[t]} p_n e^{nh(t)}} \\
&\leq \frac{1}{\Phi(t)-\Phi(M)-1} \quad \to 0, \quad (M \to \infty),
\end{aligned}
$$

i.e (2.4) is valid.

ad (2.5) :

Consider the sequence $l(n)$, satisfying

$$
\sum_{k=n}^{n+l(n)-1} \frac{p_k}{\Delta_k} < 1 \leq \sum_{k=n}^{n+l(n)} \frac{p_k}{\Delta_k} < 2.
$$

and put $n_0 = [t]-1, n_{\nu+1} = n_\nu + l(n_\nu)$; $\nu = 0, 1, 2, \ldots$ and $\alpha = \alpha(t, N) \in I\!N$, s.t. $n_{\alpha+1} > N \geq n_\alpha$ for given $t > N \in I\!N$.
Then we obtain

$$
\Phi(N) - \Phi(t) \leq \sum_{n=[t]+1}^{N} \frac{p_n}{\Delta_n} \leq \sum_{\nu=0}^{\alpha} \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n}{\Delta_n} \leq 2(\alpha+1).
$$

In (2.5) we assume that $\Phi(N) - \Phi(t) \to \infty$, $(t \to \infty)$, hence $\alpha \to \infty$, $(t \to \infty)$. Moreover using the definition and the monotonicity of $h(.)$ and (1.6), we obtain for $[t]-1 \leq s$:

$$
\begin{aligned}
\sum_{n=s+1}^{\infty} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \sum_{k=s+1}^{n} \frac{p_k}{\Delta_k} &= \sum_{k=s+1}^{\infty} \frac{p_k}{\Delta_k} \sum_{n=k}^{\infty} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \\
&= \frac{1}{p\left(e^{h(t)}\right)} \sum_{k=s+1}^{\infty} \frac{p_k}{\Delta_k} \left(\frac{e^{h(t)}}{e^{h(k)}}\right)^k \sum_{n=k}^{\infty} p_n e^{nh(k)} \left(\frac{e^{h(k)}}{e^{h(t)}}\right)^{k-n} \\
&\leq \frac{1}{p\left(e^{h(t)}\right)} \sum_{k=s+1}^{\infty} p_k e^{kh(t)} \frac{p\left(e^{h(k)}\right) e^{-kh(k)}}{\Delta_k} \\
&= \sum_{k=s+1}^{\infty} \frac{p_k e^{kh(t)}}{p\left(e^{h(t)}\right)}.
\end{aligned}
$$

Choosing especially $s=[t]-1$ and n_ν as above, we find by the choice of the n_ν:

$$\begin{aligned}
1 \geq \sum_{n=[t]+1}^{\infty} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \sum_{k=[t]+1}^{n} \frac{p_k}{\Delta_k} &= \sum_{\nu=0}^{\infty} \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \sum_{k=n_0+1}^{n} \frac{p_k}{\Delta_k} \\
&\geq \sum_{\nu=0}^{\infty} \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \sum_{k=n_0+1}^{n_\nu} \frac{p_k}{\Delta_k} \\
&\geq \sum_{\nu=0}^{\infty} \nu \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)}.
\end{aligned}$$

Using the last two chains of inequalities, we conclude for $s=N$:

$$\begin{aligned}
\sum_{n=N}^{\infty} c_n(t)(\Phi(n)-\Phi(N)) &= \sum_{n=N+1}^{\infty} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \sum_{k=N+1}^{n} \frac{p_k}{\Delta_k} \\
&\leq \sum_{n=N+1}^{\infty} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \leq \sum_{\nu=\alpha}^{\infty} \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \\
&\leq \sum_{\nu=\alpha}^{\infty} \frac{\nu}{\alpha} \sum_{n=n_\nu+1}^{n_{\nu+1}} \frac{p_n e^{nh(t)}}{p\left(e^{h(t)}\right)} \\
&\leq \frac{1}{\alpha} \to 0, \quad (t\to\infty).
\end{aligned}$$

ad (2.6):

By assumption $(1.9'')$ there exists some $c>0$ s.t.

$$s_{m+1}-s_n > -c \quad \text{for all} \quad m\in[n,n+l_n],$$

and n large enough.
With $s(t)=s_n, l(t)=l_n$ if $n\leq t<n+1, n\in I\!N_0$, we obtain

$$s(u)-s(t) > -c \;\text{ for }\; t\leq u\leq [t]+l(t),$$

hence

$$s(u)-s(t) > -2c \;\text{ for }\; t\leq u\leq t+l(t).$$

For given $u>t$ we define $t_0=t,\ \ t_{\nu+1}=t_\nu+l(t_\nu)$ with $\nu=0,1,\ldots,r$, and $r\in I\!N$ s.t. $t_{r+1}>u\geq t_r$ and obtain

$$s(u)-s(t)=\sum_{\nu=0}^{r-1}\{s(t_{\nu+1})-s(t_\nu)\}+s(u)-s(t_r)\geq -(r+1)2c.$$

By the definition of the sequence l_n, see $(1.8'')$, there exists some constant $\delta>0$, s.t.

$$0<\delta\leq \sum_{k=n}^{n+l_n} \frac{p_k}{\Delta_k}$$

for all large n and therefore

$$\begin{aligned} r &\le \frac{1}{\delta}\sum_{\nu=0}^{r-1}\sum_{k=[t_\nu]+1}^{[t_{\nu+1}]}\frac{p_k}{\Delta_k} = \frac{1}{\delta}\sum_{\nu=0}^{r-1}\{\Phi([t_{\nu+1}]) - \Phi([t_\nu])\} \\ &= \frac{1}{\delta}\{\Phi([t_r]) - \Phi([t_0])\} \le \frac{1}{\delta}\{\Phi(u) - \Phi(t) + 2\}, \end{aligned}$$

by (2.3) and the monotonicity of $\Phi(.)$.
Altogether we obtain for $u > t$ (large enough)

$$s(u) - s(t) \ge -2c(r+1) \ge -2\frac{c}{\delta}\{\Phi(u) - \Phi(t) + 2\} - 2c,$$

i.e. $a = \dfrac{2c}{\delta}, b = \dfrac{4c}{\delta} + 2c$.
Hence Lemma 1 implies Theorem 1.

3 A one–sided Tauberian Theorem

Sharpening the Tauberian–condition of Theorem 1 we get a one–sided Tauberian theorem.

THEOREM 2.
Given a regular power–series method (P) with a sequence of weights (p_n) satisfying (1.1),(1.3) and (1.4). Suppose further that (s_n) is a sequence of reals satisfying

$$s_n \to s(P) \text{ and the Tauberian–condition } (1.9'),$$

then we have $s_n \to s$.

We only sketch the **Proof:**
By Theorem 1 we find that $s_n = \boldsymbol{O}(1)$.
The rest of the proof follows by techniques used elsewhere repeatedly so we only outline the main points. W.l.o.g we assume $R = 1$ or $R = \infty$.
(a) In case $R = 1$ and $\dfrac{\log P_n}{\log n} = \boldsymbol{O}(1)$, $s_n = \boldsymbol{O}(1)$ is a Tauberian condition for $s_n \to s(P)$ to $s_n \to s(M_p)$, i.e.

$$\frac{\sum\limits_{k=0}^{n} s_k p_k}{\sum\limits_{k=0}^{n} p_k} \to s$$

(see [10,20], observe that P_n is regularly varying under (1.4)). Next $s_n \to s(M_p)$ and $(1.9')$ imply $s_n \to s$, see [6] Theorem 6.1 or [18] Theorem 3.1.

(b) In the other cases, we can choose $l(n,\lambda) = (\lambda - 1)/\sqrt{|g''(n)|}$. We are now in the situation discussed in [11] $R = 1$ and [12] $R = \infty$ and we conclude as therein that $s_n \to s(P)$ and $s_n = \boldsymbol{O}(1)$ imply that $s_n \to s(P_\alpha)$ for all $\alpha > -1$, where (P_α) is the power series method with weights $p_n(\alpha) = p_n \exp\{\alpha g(n)\}$ in case $R = 1$ and $p_n(\alpha) = p_n \exp\{-\alpha g(n)\}$ in case $R = \infty$ respectively. Using the asymptotic calculations in [11] and [12] we obtain with $s(t) = \begin{cases} s_n & t = n \in I\!N_0 \\ 0 & t < 0 \end{cases}$

and linear elsewhere, that

$$(3.2) \qquad \sqrt{\frac{|g''(t)|(\alpha+1)}{2\pi}} \int\limits_{-\infty}^{\infty} \exp\{-(\alpha+1)|g''(t)|x^2\} s(x+t)\, dx \to s, \quad (t \to \infty).$$

We may now put w.l.o.g. $s = 0$ and assume that $s_n \not\to 0$, i.e. there exists some sequence n_k s.t. $s_{n_k} \to \delta > 0$. By $(1.9')$ we find that for $\lambda - 1 > 0$, but small enough, k large

$$s(n_k + x) \geq \frac{\delta}{2} \text{ for } |x| \leq (\lambda - 1)/\sqrt{|g''(n_k)|}.$$

But since the integral

$$\sqrt{\frac{|g''(t)|(\alpha+1)}{2\pi}} \int_{|x| \leq \frac{(\lambda-1)\sqrt{\alpha+1}}{\sqrt{(\alpha+1)|g''(t)|}}} \exp\{-(\alpha+1)|g''(t)|x^2\}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{|x| \leq (\lambda-1)\sqrt{\alpha+1}} \exp\{-\frac{x^2}{2}\}\, dx \to 1, \quad (\alpha \to \infty)$$

uniformly in t, we obtain a contradiction to (3.2) and hence Theorem 2.

Remark 2:

(i) In case $\dfrac{\log P_n}{\log n} = \boldsymbol{O}(1)$ one can generalize the result to a somewhat larger class $\{p_n\}$ than considered, see [18] for details.

(ii) With our Theorem 2 we obtain all one–sided oscillation Tauberian theorems known in the literature for the special cases we gave in Remark 1.

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