Asymptotic properties of estimators for the volume fractions of jointly stationary random sets

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In the present paper, we show how a consistent estimator can be derived for the asymptotic covariance matrix of stationary 0–1–valued vector fields in $\mathbb{R}^d$, whose supports are jointly stationary random closed sets. As an example, which is of particular interest for statistical applications, we consider jointly stationary random closed sets associated with the Boolean model in $\mathbb{R}^d$ such that the components indicate the frequency of coverage by the single grains of the Boolean model. For this model, a representation formula for the entries of the covariance matrix is obtained.

Key Words and Phrases: Stationary vector field; covariance matrix; estimation; consistency; Boolean model; frequency of coverage.

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1 Preliminaries

In the following we recall some basic notions and results from stochastic geometry. More details on this can be found, e.g., in STOYAN, KENDALL and MECKE (1995), CRESSIE (1993), OHSER and MÜCKLICH (2000), and MOLCHANOV (1997). Let $\xi(x) = (\xi_1(x), \ldots, \xi_r(x))^\top$, $x \in \mathbb{R}^d$, be a stationary r–dimensional vector field in $\mathbb{R}^d$ such that the components $\xi_k(x)$, $1 \le k \le r$, are given by

$$\xi_k(x) = \mathbb{1}_{\Xi_k}(x)\,, \quad 1 \le k \le r\,, \tag{1}$$

where $\Xi_1, \ldots, \Xi_r$ are jointly stationary random closed sets (RACS) in $\mathbb{R}^d$ and $\mathbb{1}_A$ denotes the indicator function of a Borel set $A \subset \mathbb{R}^d$.

Notice that for each pair k, l with $1 \le k, l \le r$, the covariance

$$\mathrm{Cov}(\xi_k(x), \xi_l(y)) = \mathbb{E}\,\xi_k(x)\xi_l(y) - \mathbb{E}\,\xi_k(x)\,\mathbb{E}\,\xi_l(y)\,, \quad x, y \in \mathbb{R}^d,$$

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is a function of $y - x$. We thus write

$$\mathrm{Cov}_{kl}(h) = \mathrm{Cov}(\xi_k(o), \xi_l(h)), \qquad h \in \mathbb{R}^d,$$

where $o \in \mathbb{R}^d$ denotes the origin. Furthermore, suppose that the *volume fraction*

$$p_k = \mathbb{P}(o \in \Xi_k) \tag{2}$$

of the RACS Ξ_k is (hypothetically) given for each $k = 1, \ldots, r$, where $0 < p_k < 1$. Then, we we have

$$\mathrm{Cov}_{kl}(h) = \mathbb{P}(o \in \Xi_k, h \in \Xi_l) - p_k p_l, \quad 1 \leq k, l \leq r. \tag{3}$$

Notice that $\mathrm{Cov}_{kl}(-h) = \mathrm{Cov}_{lk}(h)$. In the following, the function

$$C_{kl}(h) = \mathbb{P}(o \in \Xi_k, h \in \Xi_l), \quad h \in \mathbb{R}^d, \tag{4}$$

will be called the *cross–covariance* of Ξ_k and Ξ_l. For $k = l$ we get in particular

$$C_k(h) = \mathbb{P}(o \in \Xi_k, h \in \Xi_k),$$

i.e. the *covariance* of the stationary RACS Ξ_k. Similar to (3), the *centered covariance function* $\mathrm{Cov}_k(h), h \in \mathbb{R}^d$, of Ξ_k is given by $\mathrm{Cov}_k(h) = C_k(h) - {p_k}^2$, $1 \leq k \leq r$. Notice that $\mathrm{Cov}_k(h)$, $h \in \mathbb{R}^d$, coincides with the covariance function of the stationary random field $\{\xi_k(x), x \in \mathbb{R}^d\}$, i.e.,

$$\mathrm{Cov}_k(h) = \mathrm{Cov}(\xi_k(o), \xi_k(h)). \tag{5}$$

Using the stationarity of Ξ_k, $1 \leq k \leq r$, we obtain for any Borel set $A \subset \mathbb{R}^d$ with positive and finite Lebesgue measure $|A|$ and for each $h \in \mathbb{R}^d$ that

$$C_k(h) = \frac{\mathbb{E}\,|\Xi_k \cap (\Xi_k + h) \cap A|}{|A|}.$$

Accordingly, since Ξ_k and Ξ_l are jointly stationary RACS for each $1 \leq k, l \leq r$, we have

$$C_{kl}(h) = \frac{\mathbb{E}\,|\Xi_l \cap (\Xi_k + h) \cap A|}{|A|}.$$

2 Motivation from image analysis

In this section, the consistent estimation of the covariances $\mathrm{Cov}_{kl}(h)$, $1 \leq k, l \leq r$, given in (3) will be motivated by considering an asymptotic test for the r–dimensional vector $(p_1, \ldots, p_r)^\top$ of (hypothetically given) volume fractions. Suppose that an image with r different phases is observed within a given (Borel measurable) *sampling window* $W \subset \mathbb{R}^d$ satisfying $0 < |W| < \infty$, where the phases are visualized as different tones of a grayscale image. Furthermore, we assume that the r different phases can be described by realizations of the stationary RACS Ξ_k, $1 \leq k \leq r$, which are observed within the sampling window W. Then, the *empirical volume fraction* $\widehat{p}_{W,k}$ is given by

$$\widehat{p}_{W,k} = \frac{|\Xi_k \cap W|}{|W|}, \quad 1 \leq k \leq r.$$

The covariances $\mathrm{Cov}_{kl}(h)$, $1 \le k, l \le r$, are of particular interest for the construction of an asymptotic test (based on a single observation of the r different grayscales in a large W) checking whether or not the r–dimensional vector $(p_1, \dots, p_r)^\top$ is in accordance with the observed image. Such an asymptotic test can be performed using a multivariate central limit theorem for the r–dimensional random vectors $(\widehat{p}_{W_n,1}, \dots, \widehat{p}_{W_n,r})^\top$, where W_n, $n \in \mathbb{N}$, denotes an increasing sequence of convex, compact sets in $\mathbb{R}^d$ with unboundedly growing inball radius $\rho(W_n)$. Indeed, under appropriate moment and mixing conditions imposed on the stationary RACS Ξ_k, $1 \le k \le r$, the following central limit theorem for the vector $(\widehat{p}_{W_n,1}, \dots, \widehat{p}_{W_n,r})^\top$ can be proved:

$$Y_{n,r} = \begin{pmatrix} \sqrt{|W_n|}\,(\widehat{p}_{W_n,1} - p_1) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{p}_{W_n,r} - p_r) \end{pmatrix} \underset{n\to\infty}{\overset{d}{\Longrightarrow}} N(0, \Sigma_r)\,; \tag{6}$$

see, for example, HEINRICH (2004) and MASE (1982). Related central limit theorems for general random fields with statistical applications can be found e.g. in GUYON (1995), IVANOV and LEONENKO (1989).

The covariance matrix $\Sigma_{n,r}$ of the random vector $Y_{n,r}$ is given by

$$\Sigma_{n,r} = |W_n| \begin{pmatrix} \mathrm{Var}\,\widehat{p}_{W_n,1} & \cdots & \mathrm{Cov}(\widehat{p}_{W_n,1}\,,\,\widehat{p}_{W_n,r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(\widehat{p}_{W_n,r}\,,\,\widehat{p}_{W_n,1}) & \cdots & \mathrm{Var}\,\widehat{p}_{W_n,r} \end{pmatrix}.$$

Notice that the entries of $\Sigma_{n,r}$ can be written as

$$\mathrm{Var}\,\widehat{p}_{W_n,k} = \frac{1}{|W_n|^2} \int\limits_{\mathbb{R}^d} \gamma_{W_n}(h)(C_k(h) - {p_k}^2)\,dh\,, \quad 1 \le k \le r\,, \tag{7}$$

and

$$\mathrm{Cov}(\widehat{p}_{W_n,k}\,,\,\widehat{p}_{W_n,l}) = \frac{1}{|W_n|^2} \int\limits_{\mathbb{R}^d} \gamma_{W_n}(h)(C_{kl}(h) - p_k p_l)\,dh\,, \quad 1 \le k, l \le r\,, \tag{8}$$

where $\gamma_{W_n}(h) = |W_n \cap (W_n + h)|$ denotes the *set–covariance function* of the sampling window W_n. Thus, assuming that $\int_{\mathbb{R}^d} |C_{kl}(h) - p_k p_l|\,dh < \infty$ for $1 \le k, l \le r$, the asymptotic covariance matrix $\Sigma_r = \lim_{n\to\infty} \Sigma_{n,r}$ equals

$$\Sigma_r = \begin{pmatrix} \int_{\mathbb{R}^d} (C_1(h) - {p_1}^2)\,dh & \cdots & \int_{\mathbb{R}^d} (C_{1r}(h) - p_1 p_r)\,dh \\ \vdots & \ddots & \vdots \\ \int_{\mathbb{R}^d} (C_{r1}(h) - p_r p_1)\,dh & \cdots & \int_{\mathbb{R}^d} (C_r(h) - {p_r}^2)\,dh \end{pmatrix},$$

Usually the covariance $C_k(h)$ of the RACS Ξ_k appearing in (7) and the cross–covariance $C_{kl}(h)$ of Ξ_k and Ξ_l in (8) are unknown, which prevents an exact evaluation of the entries of the covariance matrix $\Sigma_{n,r}$. Therefore, we need a consistent estimator for the asymptotic covariance matrix Σ_r. Then, Slutsky–type arguments can be employed in order to establish an asymptotic test for the r–dimensional vector $(p_1, \ldots, p_r)^\top$. Such consistent estimators can be obtained by considering the spectral representations of $C_k(h)$ and $C_{kl}(h)$ and evaluating a consistent kernel estimator of the corresponding spectral density from the observed image; see Böhm, Heinrich and Schmidt (2004). On the other hand, unbiased estimators for $C_k(h)$ and $C_{kl}(h)$, which are given by

$$\widehat{C}_{W_n,k}(h) = \frac{|\Xi_k \cap (\Xi_k + h) \cap W_n \cap (W_n + h)|}{|W_n \cap (W_n + h)|}, \quad 1 \le k \le r,$$

and

$$\widehat{C}_{W_n,kl}(h) = \frac{|\Xi_l \cap (\Xi_k + h) \cap W_n \cap (W_n + h)|}{|W_n \cap (W_n + h)|}, \quad 1 \le k, l \le r, \tag{9}$$

(provided that $\gamma_{W_n}(h) > 0$) can be used in order to get a consistent estimator for Σ_r.

3 Consistent estimator of the covariance matrix

In the following we will specify conditions on the r–dimensional vector field given in Section 1, such that the matrix

$$\widehat{\Sigma}_{n,r} = \Big(\frac{1}{|W_n|} \int_{V_n} \gamma_{W_n}(h) (\widehat{C}_{W_n,kl}(h) - \widehat{p}_{W_n,k}\, \widehat{p}_{W_n,l})\, dh \Big)_{k,l=1}^{r}, \quad V_n \subset W_n \tag{10}$$

is an asymptotically unbiased and mean–square consistent estimator for Σ_r, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\, \widehat{\Sigma}_{n,r} = \Sigma_r \quad \text{or, equivalently,} \quad \lim_{n\to\infty} ||\mathbb{E}\, \widehat{\Sigma}_{n,r} - \Sigma_r|| = 0 \tag{11}$$

and, moreover,

$$\lim_{n\to\infty} \mathbb{E}\, ||\widehat{\Sigma}_{n,r} - \Sigma_r||^2 = 0\,, \tag{12}$$

where $||A|| = \Big(\sum_{k,l=1}^{r} a_{kl}^2 \Big)^{1/2}$ for some matrix $A = \Big(a_{kl} \Big)_{k,l=1}^{r}$.

To state our result we introduce the mixed cumulant $\Gamma_k(Z_1, \ldots, Z_k)$ of order k of the random vector $(Z_1, \ldots, Z_k)^\top \in \mathbb{R}^k$, which is defined by

$$\Gamma_k(Z_1, \ldots, Z_k) = \frac{1}{i^k} \frac{d^k}{dt_1 \cdots dt_k} \log \mathbb{E}\, \mathrm{e}^{i(t_1 Z_1 + \ldots + t_k Z_k)} \Big|_{t_1 = \ldots = t_k = 0}$$

if $\mathbb{E}\, |Z_i|^k < \infty$ for each $1 \le i \le k$. Using $\overline{Z}_i = Z_i - \mathbb{E}\, Z_i$ we obtain for $k = 2, 3, 4$ that

$$\Gamma_2(Z_1, Z_2) = \mathbb{E}\, \overline{Z}_1 \overline{Z}_2 = \mathrm{Cov}(Z_1, Z_2)\,, \qquad \Gamma_3(Z_1, Z_2, Z_3) = \mathbb{E}\, \overline{Z}_1 \overline{Z}_2 \overline{Z}_3\,,$$

$$\Gamma_4(Z_1, Z_2, Z_3, Z_4) = \mathbb{E}\, \overline{Z}_1 \overline{Z}_2 \overline{Z}_3 \overline{Z}_4 - \mathbb{E}\, \overline{Z}_1 \overline{Z}_2 \mathbb{E}\, \overline{Z}_3 \overline{Z}_4 - \mathbb{E}\, \overline{Z}_1 \overline{Z}_3 \mathbb{E}\, \overline{Z}_2 \overline{Z}_4 - \mathbb{E}\, \overline{Z}_1 \overline{Z}_4 \mathbb{E}\, \overline{Z}_2 \overline{Z}_3\,.$$

Furthermore, we have

$$\Gamma_k(Z_{\pi(1)}, \ldots, Z_{\pi(k)}) = \Gamma_k(Z_1, \ldots, Z_k)$$

for any permutation π of the set $\{1,\ldots,k\}$. We will use the following notation for any $k,l\in\{1,\ldots,r\}$.

$$
\begin{aligned}
c_{kl}^{(1,1)}(x) &= \Gamma_2(\xi_k(o),\xi_l(x)) = \mathrm{Cov}(\xi_k(o),\xi_l(x))\,,\\
c_{kl}^{(1,2)}(x,y) &= \Gamma_3(\xi_k(o),\xi_l(x),\xi_l(y))\,,\\
c_{kl}^{(2,2)}(x,y,z) &= \Gamma_4(\xi_k(o),\xi_k(x),\xi_l(y),\xi_l(z))\,.
\end{aligned}
$$

Theorem 3.1 *Let $(\Xi_1,\ldots,\Xi_r)^\top$ be a vector of jointly stationary RACS in $\mathbb{R}^d$ and W_n, $n\in\mathbb{N}$, an increasing sequence of convex and compact sampling windows in $\mathbb{R}^d$ with $\lim_{n\to\infty}\rho(W_n)=\infty$, where $\rho(W_n)=\sup\{r\geq 0 : b(x,r)\subseteq W_n,\, x\in W_n\}$ and $b(x,r)$ denotes the disk with radius r that is centered at $x\in\mathbb{R}^d$. Furthermore, let $V_n=b(o,\varepsilon_n\sqrt{\rho(W_n)})$ and $\varepsilon_n\downarrow 0$ a sequence such that $\lim_{n\to\infty}\varepsilon_n^2\rho(W_n)=\infty$. Assume that for any $k,l\in\{1,\ldots,r\}$ and $n\in\mathbb{N}$*

$$
\int_{\mathbb{R}^d}|c_{kl}^{(1,1)}(x)|\,dx<\infty \tag{13}
$$

and

$$
\frac{1}{|V_n|^2}\int_{V_n}\int_{V_n}\int_{\mathbb{R}^d}\big|\,\mathbb{P}(o\in\Xi_l\cap(\Xi_k+x),z\in\Xi_l\cap(\Xi_k+y))
$$
$$
-\,\mathbb{P}(o\in\Xi_l\cap(\Xi_k+x))\;\mathbb{P}(o\in\Xi_l\cap(\Xi_k+y))\big|\,dz\,dy\,dx\leq c_0<\infty\,. \tag{14}
$$

Then, (11) *and* (12) *hold, i.e., $\widehat{\Sigma}_{n,r}$ is an asymptotically unbiased and mean-square consistent estimator for Σ_r.*

Proof Exploiting the fact that W_n is a convex body we get

$$
|W_n|-|W_n\cap(W_n+h)|\leq|\{x\in W_n : b(x,\|h\|)\cap W_n^c\neq\emptyset\}|\leq\|h\|\,H_{d-1}(\partial W_n)
$$

for any $h\in b(o,\rho(W_n))$, where $H_{d-1}(\partial W_n)$ denotes the $(d-1)$–dimensional Hausdorff measure of the surface $\partial W_n\subset\mathbb{R}^d$; see Heinrich and Pawlas (2004) for details. Together with the well–known inequality

$$
H_{d-1}(\partial W_n)\,\rho(W_n)\leq d\,|W_n|
$$

proved by Wills (1970), we arrive at

$$
1-\frac{|W_n\cap(W_n+h)|}{|W_n|}\leq\frac{d\,\|h\|}{\rho(W_n)}\quad\text{for}\quad h\in b(o,\rho(W_n))\,.
$$

Thus,

$$
\sup_{h\in V_n}\left|1-\frac{\gamma_{W_n}(h)}{|W_n|}\right|\underset{n\to\infty}{\longrightarrow}0\,.
$$

Therefore, in order to prove (11), we show that for each $1\leq k,l\leq r$

$$
\lim_{n\to\infty}\mathbb{E}\int_{V_n}(\widehat{C}_{W_n,kl}(h)-\widehat{p}_{W_n,k}\,\widehat{p}_{W_n,l})\,dh=\int_{\mathbb{R}^d}(C_{kl}(h)-p_kp_l)\,dh\,. \tag{15}
$$

Since $\mathbb{E}\,\widehat{C}_{W_n,kl}(h) = C_{kl}(h)$ and $\mathbb{E}\,\widehat{p}_{W_n,k} = p_k$ we have

$$\mathbb{E}\int_{V_n} (\widehat{C}_{W_n,kl}(h) - \widehat{p}_{W_n,k}\,\widehat{p}_{W_n,l})\,dh = \int_{V_n} (C_{kl}(h) - p_k p_l)\,dh$$
$$-|V_n|\,\mathbb{E}\,(\widehat{p}_{W_n,k} - p_k)(\widehat{p}_{W_n,l} - p_l)\,.$$

Using

$$\Big|\int_{V_n^c} (C_{kl}(h) - p_k p_l)\,dh\Big| \le \int_{V_n^c} |c_{kl}^{(1,1)}(x)|\,dx \underset{n\to\infty}{\longrightarrow} 0\,,$$

$$\begin{aligned}
&|V_n|\,\mathbb{E}\,(\widehat{p}_{W_n,k} - p_k)(\widehat{p}_{W_n,l} - p_l)\\
&= \frac{|V_n|}{|W_n|^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1\!\!1_{W_n}(x)\,1\!\!1_{W_n}(y)\,\Gamma_2(\xi_k(x),\xi_l(y))\,dx\,dy\\
&= \frac{|V_n|}{|W_n|^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1\!\!1_{W_n}(x)\,1\!\!1_{W_n}(z+x)\,c_{kl}^{(1,1)}(z)\,dx\,dz \le \frac{|V_n|}{|W_n|}\int_{\mathbb{R}^d} |c_{kl}^{(1,1)}(z)|\,dz\,,
\end{aligned}$$

and $|W_n| \ge |b(o,\rho(W_n))|$ we obtain (15). Now, we prove (12) by showing that for each $1 \le k,l \le r$

$$\lim_{n\to\infty}\mathbb{E}\Big(\int_{V_n} (\widehat{C}_{W_n,kl}(h) - \widehat{p}_{W_n,k}\,\widehat{p}_{W_n,l})\,dh - \int_{\mathbb{R}^d} (C_{kl}(h) - p_k p_l)\,dh\Big)^2 = 0\,.$$

This is satisfied whenever

$$\lim_{n\to\infty}\mathbb{E}\Big(\int_{V_n} (\widehat{C}_{W_n,kl}(h) - C_{kl}(h))\,dh\Big)^2 = 0\,,$$

since

$$\Big|\int_{V_n^c} (C_{kl}(h) - p_k p_l)\,dh\Big| \underset{n\to\infty}{\longrightarrow} 0 \quad\text{and}\quad |V_n|^2\,\mathbb{E}\,(\widehat{p}_{W_n,k}\,\widehat{p}_{W_n,l} - p_k p_l)^2 \underset{n\to\infty}{\longrightarrow} 0\,,$$

where the latter relation is valid because of

$$\frac{|V_n|^2}{|W_n|} = \frac{|b(o,1)|^2\varepsilon_n^{2d}(\rho(W_n))^d}{|W_n|} \le |b(o,1)|\,\varepsilon_n^{2d} \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\begin{aligned}
\mathbb{E}\,(\widehat{p}_{W_n,k}\,\widehat{p}_{W_n,l} - p_k p_l)^2 &= \mathbb{E}\,\Big((\widehat{p}_{W_n,k} - p_k)\widehat{p}_{W_n,l} + p_k(\widehat{p}_{W_n,l} - p_l)\Big)^2\\
&\le 2\,\mathbb{E}\,(\widehat{p}_{W_n,k} - p_k)^2 + 2\,\mathbb{E}\,(\widehat{p}_{W_n,l} - p_l)^2\\
&\le \frac{2}{|W_n|}\int_{\mathbb{R}^d} |c_{kk}^{(1,1)}(x)|\,dx + \frac{2}{|W_n|}\int_{\mathbb{R}^d} |c_{ll}^{(1,1)}(x)|\,dx\,.
\end{aligned}$$

Using the definition of $\widehat{C}_{W_n,kl}(h)$ and $C_{kl}(h)$ we obtain that

$$\widehat{C}_{W_n,kl}(h) - C_{kl}(h) = \frac{1}{|W_{n,h}|}\int_{W_{n,h}} \Big(1\!\!1_{\Xi_l\cap(\Xi_k+h)}(u) - \mathbb{E}\ 1\!\!1_{\Xi_l\cap(\Xi_k+h)}(u)\Big)\,du\,,$$

where $W_{n,h} = W_n \cap (W_n + h)$. This gives

$$\begin{aligned}
&\mathbb{E}\Big(\int_{V_n} (\widehat{C}_{W_n,kl}(h) - C_{kl}(h))\, dh \Big)^2 \\
&= \int_{V_n}\int_{V_n} \mathbb{E}\,(\widehat{C}_{W_n,kl}(x) - C_{kl}(x))(\widehat{C}_{W_n,kl}(y) - C_{kl}(y))\, dx\, dy \\
&= \int_{V_n}\int_{V_n} \frac{dx\, dy}{|W_{n,x}|\,|W_{n,y}|} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1\!\!1_{W_{n,x}}(u)\ 1\!\!1_{W_{n,y}}(v) \\
&\qquad\qquad \times \operatorname{Cov}\Big(1\!\!1_{\Xi_l \cap (\Xi_k + x)}(u),\ 1\!\!1_{\Xi_l \cap (\Xi_k + y)}(v) \Big)\, du\, dv \\
&= \int_{V_n}\int_{V_n}\int_{\mathbb{R}^d} \frac{|W_n \cap (W_n + x) \cap (W_n - z) \cap (W_n + y - z)|}{|W_n \cap (W_n + x)|\,|W_n \cap (W_n + y)|} \\
&\quad \times \operatorname{Cov}\Big(1\!\!1_{\Xi_l \cap (\Xi_k + x)}(o),\ 1\!\!1_{\Xi_l \cap (\Xi_k + y)}(z) \Big)\, dz\, dy\, dx\,.
\end{aligned}$$

Using (14) we have

$$\mathbb{E}\Big(\int_{V_n} (\widehat{C}_{W_n,kl}(h) - C_{kl}(h))\, dh \Big)^2 \le c_0\, \frac{|V_n|^2}{\inf\limits_{x \in V_n} |W_n \cap (W_n + x)|}$$

and consequently, by $|W_n \cap (W_n + x)| \ge |b(o, \rho(W_n) - \|x\|)|$,

$$\frac{|V_n|^2}{\inf\limits_{x \in V_n} |W_n \cap (W_n + x)|} \le |b(o,1)| \Big(\frac{\varepsilon_n^2}{1 - \varepsilon_n / \sqrt{\rho(W_n)}} \Big)^d \underset{n \to \infty}{\longrightarrow} 0\,,$$

which completes the proof.

Notice that assumption (14) can be replaced by certain conditions on $c_{kl}^{(2,2)}$, $c_{kl}^{(1,2)}$, and $c_{lk}^{(1,2)}$ if the following decomposition is considered:

$$\begin{aligned}
&\operatorname{Cov}\Big(1\!\!1_{\Xi_l \cap (\Xi_k + x)}(o),\ 1\!\!1_{\Xi_l \cap (\Xi_k + y)}(z) \Big) \\
&= \operatorname{Cov}\Big((1\!\!1_{\Xi_l}(o) - p_l)(1\!\!1_{\Xi_k + x}(o) - p_k),\ (1\!\!1_{\Xi_l}(z) - p_l)(1\!\!1_{\Xi_k + y}(z) - p_k) \Big) \\
&\quad + \mathbb{E}\Big(p_k (1\!\!1_{\Xi_l}(o) - p_l) + p_l (1\!\!1_{\Xi_k + x}(o) - p_k) \Big) (1\!\!1_{\Xi_l}(z) - p_l)(1\!\!1_{\Xi_k + y}(z) - p_k) \\
&\quad + \mathbb{E}\,(1\!\!1_{\Xi_l}(o) - p_l)(1\!\!1_{\Xi_k + x}(o) - p_k) \Big(p_k (1\!\!1_{\Xi_l}(z) - p_l) + p_l (1\!\!1_{\Xi_k + y}(z) - p_k) \Big) \\
&\quad + \mathbb{E}\Big(p_k (1\!\!1_{\Xi_l}(o) - p_l) + p_l (1\!\!1_{\Xi_k + x}(o) - p_k) \Big) \\
&\qquad\qquad \times \Big(p_k (1\!\!1_{\Xi_l}(z) - p_l) + p_l (1\!\!1_{\Xi_k + y}(z) - p_k) \Big).
\end{aligned}$$

Using the stationarity of the random field $(\xi_k(x), \xi_l(x))^\top$, $x \in \mathbb{R}^d$, we get

$$\begin{aligned}
&\operatorname{Cov}\Big(1\!\!1_{\Xi_l \cap (\Xi_k + x)}(o),\ 1\!\!1_{\Xi_l \cap (\Xi_k + y)}(z) \Big) \\
&= c_{kl}^{(2,2)}(z + x - y, z + x, x) + c_{kk}^{(1,1)}(z + x - y)\, c_{ll}^{(1,1)}(z) + c_{kl}^{(1,1)}(y - z)\, c_{kl}^{(1,1)}(z + x) \\
&\quad + p_k\, c_{kl}^{(1,2)}(y, y - z) + p_k\, c_{kl}^{(1,2)}(x, x + z) + p_l\, c_{lk}^{(1,2)}(-y, -z - x) + p_l\, c_{lk}^{(1,2)}(-x, z - y) \\
&\quad + p_k^2\, c_{ll}^{(1,1)}(z) + p_l^2\, c_{kk}^{(1,1)}(z + x - y) + p_k p_l\, c_{kl}^{(1,1)}(z - y) + p_k p_l\, c_{kl}^{(1,1)}(z + x)\,.
\end{aligned}$$

This immediately leads to the following result.

Lemma 3.1 *Assuming that* $\int_{\mathbb{R}^d} |c_{kl}^{(1,1)}(x)|\,dx < \infty$ *for any* $k,l \in \{1,\ldots,r\}$*, condition* (14) *is satisfied if*

$$\frac{1}{|V_n|^2}\int_{V_n}\int_{V_n}\int_{\mathbb{R}^d} |c_{kl}^{(2,2)}(z,y+z,x)|\,dz\,dy\,dx \le c_1 < \infty \tag{16}$$

and

$$\frac{1}{|V_n|}\int_{V_n}\int_{\mathbb{R}^d} |c_{kl}^{(1,2)}(x,z)|\,dz\,dx \le c_2 < \infty\,. \tag{17}$$

Notice that conditions (16) and (17) are satisfied whenever

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |c_{kl}^{(2,2)}(z,y,x)|\,dz\,dy\,dx + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |c_{kl}^{(1,2)}(x,z)|\,dz\,dx < \infty \tag{18}$$

or

$$\sup_{x,y\in\mathbb{R}^d}\int_{\mathbb{R}^d} |c_{kl}^{(2,2)}(z,y+z,x)|\,dz < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} |c_{kl}^{(1,2)}(x,z)|\,dz < \infty\,. \tag{19}$$

We also remark that the volume fraction of stationary RACS can be seen as one of the $d+1$ specific intrinsic volumes, or, equivalently, specific Minkowski functionals of stationary RACS in $\mathbb{R}^d$. Joint estimators for such multidimensional characteristics of stationary RACS and consistent estimators for their asymptotic covariance matrix have been investigated in Schmidt and Spodarev (2004); see also Klenk, Schmidt and Spodarev (2004) for related computational issues.

4 A multiphase Boolean model

In the following we consider stationary RACS $\Xi_1, \Xi_2, \ldots,$ which are deduced from the Boolean model in $\mathbb{R}^d$ such that the different sets indicate the frequency of coverage by the single grains of the underlying Boolean model. Notice that these RACS are not Boolean models anymore. However, for these RACS representation formulas of the quantities p_k, $C_k(h)$, and $C_{kl}(h)$ can be given. For certain grain distributions even explicit formulae can be obtained for these quantities. Therefore, the definition of the Boolean model will be briefly recalled in the following. Let

$$\Psi = \sum_{n\ge 1} \delta_{[X_n,M_n]} \sim P_{\lambda,Q}$$

be an independently marked stationary Poisson process in $\mathbb{R}^d$ with finite and positive intensity λ of the points $\{X_n,\, n\ge 1\}$ and with marks $\{M_n,\, n\ge 1\}$ being a sequence of independent copies of a non–empty, compact RACS M_0 in $\mathbb{R}^d$ (called *typical grain*), where Q denotes the distribution of the marks $\{M_n,\, n\ge 1\}$ and $P_{\lambda,Q}$ the distribution of the marked Poisson process Ψ. Furthermore, we assume that

$$\mathbb{E}\,|M_0| = \int_{\mathbb{K}} |K|\,Q(dK) < \infty\,, \tag{20}$$

where $\mathbb{K}$ denotes the space of all non-empty, compact sets in $\mathbb{R}^d$. Then, the union set

$$\Xi = \bigcup_{n\geq 1} (M_n + X_n) \tag{21}$$

is called a stationary *Boolean (Poisson–grain) model* in $\mathbb{R}^d$, where the X_n are called *germs*, and the M_n *grains.* Notice that in general condition (20) does not ensure the closeness of Ξ. However, provided that (20) holds, the union Ξ given in (21) is almost surely closed if and only if $\mathbb{E}\,|M_0 \oplus b(o,\varepsilon)| < \infty$ for some $\varepsilon > 0$; see also HEINRICH (2004) and references therein. Now we consider the random field

$$\Phi(x) = \sum_{n\geq 1} \mathbb{1}_{(M_n+X_n)}(x)\,, \quad x \in \mathbb{R}^d,$$

and the r–dimensional random vector $\xi(x) = (\xi_1(x), \ldots, \xi_r(x))^\top$, $x \in \mathbb{R}^d$, with components $\xi_k(x) = \mathbb{1}(\Phi(x) \geq k)$, $1 \leq k \leq r$, which is a stationary r–dimensional vector field in $\mathbb{R}^d$. Then, for each $1 \leq k \leq r$,

$$\Xi_k = \{x \in \Xi :\, \xi_k(x) = 1\} \tag{22}$$

is a stationary RACS in $\mathbb{R}^d$, where Ξ_k contains those areas of the Boolean model Ξ, which are covered by at least k of the shifted grains $M_n + X_n$, $n \geq 1$. Conversely, according to (1), we have the relation

$$\xi_k(x) = \mathbb{1}_{\Xi_k}(x)\,, \quad 1 \leq k \leq r\,.$$

Notice that $\Xi_1 \supseteq \Xi_2 \supseteq \ldots \supseteq \Xi_r$, where $\Xi_1 = \Xi$. This decomposition of the Boolean model Ξ into the sequence of stationary RACS $\Xi_1, \Xi_2, \ldots, \Xi_r$ will be called a *multiphase model deduced from the Boolean model Ξ with r different phases Ξ_k, $1 \leq k \leq r$.*

The decomposition of the Boolean model Ξ into the infinite sequence of stationary RACS $\Xi_1, \Xi_2, \ldots$ will just be called a *multiphase model deduced from the Boolean model* Ξ. In addition, for this model let us denote by the stationary RACS

$$\Xi^{(i)} = \Xi_i \setminus \mathrm{int}(\Xi_{i+1})\,, \quad i \geq 1\,,$$

those areas of the Boolean model Ξ, which are covered by exactly i of the shifted grains $M_n + X_n$, $n \geq 1$, where $\mathrm{int}(\Xi_{i+1})$ denotes the set of all interior points of Ξ_{i+1}.

In Figure 1, the realization of a multiphase model deduced from the Boolean model Ξ in $\mathbb{R}^2$ with two different phases Ξ_1 and Ξ_2 is given within a sampling window of 800×800 pixel points. The underlying Boolean model Ξ has intensity $\lambda = 8.8254 \cdot 10^{-5}$ and volume fraction $p = 0.5$, where the typical grain is a disc with uniformly distributed radius between 40 and 60 pixel points. Here, the multiphase model is visualized by a grayscale image, where the gray phase represents the RACS $\Xi^{(1)}$ and the black phase the RACS Ξ_2. This means that the gray phase indicates those areas of the Boolean model Ξ, which are covered by exactly one of the shifted grains $M_n + X_n$, $n \geq 1$, and the black phase those areas, which are covered by at least two of the shifted grains.

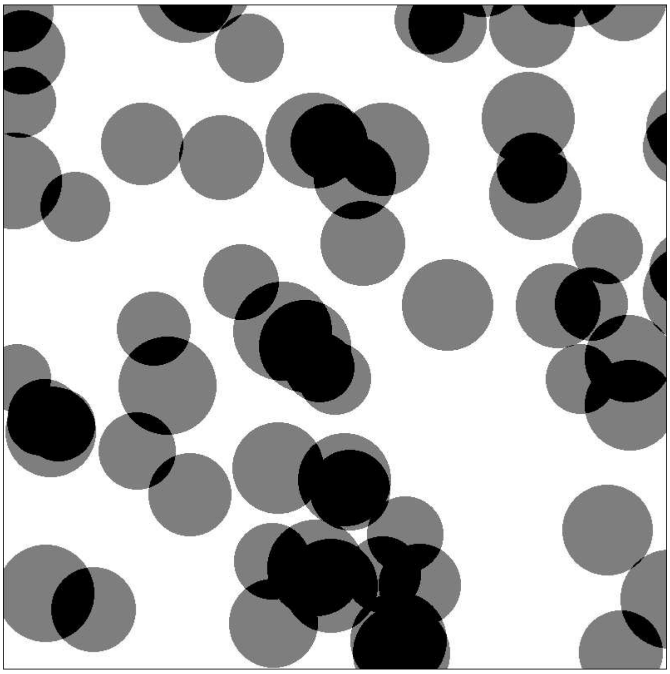

Figure 1: Realisation of a multiphase model deduced from a Boolean model in $\mathbb{R}^2$.

For the above defined multiphase model deduced from the Boolean model Ξ in $\mathbb{R}^d$, the quantities p_k and $C_{kl}(h)$ introduced in (2) and (4), respectively, can be determined as follows. Since $\Xi^{(1)}, \Xi^{(2)}, \ldots$ forms a sequence of pairwise disjoint stationary RACS in $\mathbb{R}^d$, we have

$$p_k = 1 - \sum_{i=0}^{k-1} \mathbb{P}(o \in \Xi^{(i)}) , \tag{23}$$

and

$$C_{kl}(h) = \sum_{i \geq k} \sum_{j \geq l} \mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)}) , \tag{24}$$

where $\Xi^{(0)} = \mathbb{R}^d \setminus \mathrm{int}(\Xi)$. Since the probability $\mathbb{P}(o \in \Xi^{(i)})$ can be deduced from $\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)})$, only the evaluation of $\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)})$ is required, which is the probability that the origin $o \in \mathbb{R}^d$ is covered by exactly i of the shifted grains $M_n + X_n$, $n \geq 1$, and some point $h \in \mathbb{R}^d$ is covered by exactly j of the shifted grains.

Theorem 4.1 *Let Ξ be a Boolean model in $\mathbb{R}^d$ with intensity λ and typical grain M_0 satisfying* (20). *Then, for any $h \in \mathbb{R}^d$ and $i, j \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)}) = \sum_{m=0}^{\min\{i,j\}} \frac{\lambda^{i+j-m}}{m!\,(i-m)!\,(j-m)!} \left(\mathbb{E}\,|M_0 \cap (M_0+h)|\right)^m$$
$$\times \left(\mathbb{E}\,|M_0| - \mathbb{E}\,|M_0 \cap (M_0+h)|\right)^{i+j-2m} \exp(-\lambda\,\mathbb{E}\,|M_0 \cup (M_0+h)|\,) .$$

Proof To begin with we assume that $1 \le i \le j$. Further, let $m_1, \dots, m_k, n_1, \dots, n_j \ge 1$ and $q_1, \dots, q_{i-k} \ge 1$ be pairwise different indices with $\{q_1, \dots, q_{i-k}\} \subset \{n_1, \dots, n_j\}$, where $0 \le k \le i$. Then,

$$
\begin{aligned}
&A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j} \\
&= \Big\{\, o \in M_{q_1} + X_{q_1}, \dots, o \in M_{q_{i-k}} + X_{q_{i-k}}, o \in M_{m_1} + X_{m_1}, \dots, o \in M_{m_k} + X_{m_k}, \\
&\qquad o \notin \bigcup_{l \ne q_1,\dots,q_{i-k},m_1,\dots,m_k} (M_l + X_l), \\
&\qquad h \in M_{n_1} + X_{n_1}, \dots, h \in M_{n_j} + X_{n_j}, h \notin \bigcup_{l \ne n_1,\dots,n_j} (M_l + X_l) \,\Big\}
\end{aligned}
$$

denotes the event that the origin $o \in \mathbb{R}^d$ is covered by the shifted grains $M_{q_1} + X_{q_1}, \dots, M_{q_{i-k}} + X_{q_{i-k}}, M_{m_1} + X_{m_1}, \dots, M_{m_k} + X_{m_k}$, but not touched by any other grain $M_l + X_l$, and that the vector $h \in \mathbb{R}^d$ is covered by the grains $M_{n_1} + X_{n_1}, \dots, h \in M_{n_j} + X_{n_j}$, but not touched by any other grain $M_l + X_l$, where $M_{q_1} + X_{q_1}, \dots, M_{q_{i-k}} + X_{q_{i-k}}$ represent those grains, which cover both the origin o and h. Thus, the quantity $\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)})$ equals the probability of the event

$$
B_{i,j} = \bigcup_{0 \le k \le i} \; \bigcup_{\substack{\{q_1,\dots,q_{i-k}\} \\ \subset \{n_1,\dots,n_j\}}} \; \bigcup_{\substack{1 \le m_1 < \dots < m_k \\ 1 \le n_1 < \dots < n_j}} A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j} \,,
$$

where the following two properties (i) and (ii) are valid.

(i) $A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j} = A_{q_{\pi(1)},\dots,q_{\pi(i-k)}\,;\,m_{\pi(1)},\dots,m_{\pi(k)}\,;\,n_{\pi(1)},\dots,n_{\pi(j)}}$,

(ii) $A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j} \cap A_{p'_1,\dots,p'_{i-k}\,;\,m'_1,\dots,m'_k\,;\,n'_1,\dots,n'_j} = \emptyset$,
if $\{q_1,\dots,q_{i-k}\} \times \{m_1,\dots,m_k\} \times \{n_1,\dots,n_j\} \ne \{p'_1,\dots,p'_{i-k}\} \times \{m'_1,\dots,m'_k\} \times \{n'_1,\dots,n'_j\}$.

Using properties (i) and (ii) it follows that

$$
\begin{aligned}
&\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)}) = \mathbb{P}(B_{i,j}) = \mathbb{E}\, \mathbb{1}(B_{i,j}) \\
&= \mathbb{E} \sum_{0 \le k \le i} \; \sum_{\substack{\{q_1,\dots,q_{i-k}\} \\ \subset \{n_1,\dots,n_j\}}} \; \sum_{\substack{1 \le m_1 < \dots < m_k \\ 1 \le n_1 < \dots < n_j}} \mathbb{1}(A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j}) \\
&= \mathbb{E} \sum_{0 \le k \le i} \; \sum_{\substack{\{q_1,\dots,q_{i-k}\} \\ \subset \{n_1,\dots,n_j\}}} \frac{1}{k!\, j!} \sum_{\substack{m_1,\dots,m_k \ge 1 \\ n_1,\dots,n_j \ge 1}}^{*} \mathbb{1}(A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j}) \,, \qquad (25)
\end{aligned}
$$

where the symbol $\sum^*$ means the summation over pairwise different indices. Furthermore, we have

$$
\begin{aligned}
&\mathbb{1}(A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j}) \\
&= \mathbb{1}_{\check{M}_{q_1}}(X_{q_1}) \cdot \ldots \cdot \mathbb{1}_{\check{M}_{q_{i-k}}}(X_{q_{i-k}}) \cdot \mathbb{1}_{\check{M}_{m_1}}(X_{m_1}) \cdot \ldots \cdot \mathbb{1}_{\check{M}_{m_k}}(X_{m_k})
\end{aligned}
$$

$$\prod_{l \neq q_1,\ldots,q_{i-k},m_1,\ldots,m_k} \Big(1 - \mathbb{1}_{\check{M}_l}(X_l)\Big) \cdot \mathbb{1}_{\check{M}_{n_1}+h}(X_{n_1}) \cdot \ldots \cdot \mathbb{1}_{\check{M}_{n_j}+h}(X_{n_j})$$

$$\prod_{l \neq n_1,\ldots,n_j} \Big(1 - \mathbb{1}_{\check{M}_l+h}(X_l)\Big)\,,$$

where $\check{K} = \{-x : x \in K\}$ denotes the reflection at the origin $o \in \mathbb{R}^d$. Thus, we obtain that

$$\mathbb{E} \sum^{*}_{\substack{m_1,\ldots,m_k \geq 1 \\ n_1,\ldots,n_j \geq 1}} \mathbb{1}\,(A_{q_1,\ldots,q_{i-k}\,;\,m_1,\ldots,m_k\,;\,n_1,\ldots,n_j})$$

$$= \int_{N_{\mathbb{K}}} \sum^{*}_{[x_{m_1},K_{m_1}],\ldots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\ldots,[x_{n_j},K_{n_j}] \in S_\psi} \mathbb{1}_{\check{K}_{q_1}}(x_{q_1}) \cdot \ldots \cdot \mathbb{1}_{\check{K}_{q_{i-k}}}(x_{q_{i-k}})$$

$$\mathbb{1}_{\check{K}_{m_1}}(x_{m_1}) \cdot \ldots \cdot \mathbb{1}_{\check{K}_{m_k}}(x_{m_k}) \cdot \mathbb{1}_{\check{K}_{n_1}+h}(x_{n_1}) \cdot \ldots \cdot \mathbb{1}_{\check{K}_{n_j}+h}(x_{n_j})$$

$$\prod_{[y,L] \in S_\psi \setminus \{[x_{m_1},K_{m_1}],\ldots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\ldots,[x_{n_j},K_{n_j}]\}} \Big(1 - \mathbb{1}_{\check{L}}(y)\Big)\Big(1 - \mathbb{1}_{\check{L}+h}(y)\Big)$$

$$\prod_{[y,L] \in \{[x_{n_1},K_{n_1}],\ldots,[x_{n_j},K_{n_j}]\} \setminus \{[x_{q_1},K_{q_1}],\ldots,[x_{q_{i-k}},K_{q_{i-k}}]\}} \Big(1 - \mathbb{1}_{\check{L}}(y)\Big)$$

$$\prod_{[y,L] \in \{[x_{m_1},K_{m_1}],\ldots,[x_{m_k},K_{m_k}]\}} \Big(1 - \mathbb{1}_{\check{L}+h}(y)\Big)\, P_{\lambda,Q}(d\psi)\,,$$

where $N_{\mathbb{K}}$ denotes the space of the marked counting measure $\psi = \sum_{n \geq 1} \delta_{[x_n,K_n]}$ and S_ψ the support of the marked counting measure ψ with

$$S_\psi = \{[x,K] : [x,K] \in \mathbb{R}^d \times \mathbb{K},\ \psi(\{[x,K]\}) > 0\}\,.$$

Using the refined Campbell theorem with the reduced $(k+j)-$fold Palm distribution $P^{!}_{\lambda,Q\,;\,[x_{m_1},K_{m_1}],\ldots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\ldots,[x_{n_j},K_{n_j}]}$ of the marked Poisson process Ψ (see, for example, Chapter 12 of Daley and Vere-Jones (1988)), we have

$$\mathbb{E} \sum^{*}_{\substack{m_1,\ldots,m_k \geq 1 \\ n_1,\ldots,n_j \geq 1}} \mathbb{1}\,(A_{q_1,\ldots,q_{i-k}\,;\,m_1,\ldots,m_k\,;\,n_1,\ldots,n_j})$$

$$= \int_{(\mathbb{R}^d \times \mathbb{K})^{k+j}} \prod_{[y,L] \in \{[x_{q_1},K_{q_1}],\ldots,[x_{q_{i-k}},K_{q_{i-k}}]\}} \mathbb{1}_{\check{L}}(y) \cdot \mathbb{1}_{\check{L}+h}(y)$$

$$\prod_{[y,L] \in \{[x_{m_1},K_{m_1}],\ldots,[x_{m_k},K_{m_k}]\}} \mathbb{1}_{\check{L}}(y) \cdot \Big(1 - \mathbb{1}_{\check{L}+h}(y)\Big)$$

$$\prod_{[y,L] \in \{[x_{n_1},K_{n_1}],\ldots,[x_{n_j},K_{n_j}]\} \setminus \{[x_{q_1},K_{q_1}],\ldots,[x_{q_{i-k}},K_{q_{i-k}}]\}} \Big(1 - \mathbb{1}_{\check{L}}(y)\Big) \cdot \mathbb{1}_{\check{L}+h}(y)$$

$$\int_{N_{\mathbb{K}}} \prod_{[y,L] \in S_\psi} \Big(1 - \mathbb{1}_{\check{L}}(y) - \mathbb{1}_{\check{L}+h}(y) + \mathbb{1}_{\check{L}\cap(\check{L}+h)}(y)\Big)$$

$$P^!_{\lambda,Q\,;\,[x_{m_1},K_{m_1}],\dots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\dots,[x_{n_j},K_{n_j}]}(d\psi)$$

$$\alpha^!_{k+j}\Big(d([x_{m_1},K_{m_1}],\dots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\dots,[x_{n_j},K_{n_j}])\Big)\,,$$

where $\alpha^!_{k+j}$ is the factorial moment measure of Ψ of order $k+j$. Since Ψ is an independently marked stationary Poisson process in $\mathbb{R}^d$ with intensity λ, the factorial moment measure $\alpha^!_{k+j}$ can be written as

$$\begin{aligned}\alpha^!_{k+j}\Big(d([x_{m_1},K_{m_1}],\dots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\dots,[x_{n_j},K_{n_j}])\Big) \\ = \lambda^{k+j}\,dx_{m_1}\cdots dx_{m_k}\,dx_{n_1}\cdots dx_{n_j}\,Q(dK_{m_1})\cdots Q(dK_{m_k}) \\ \times\, Q(dK_{n_1})\cdots Q(dK_{n_j})\,. \qquad (26)\end{aligned}$$

Furthermore, Slivnyak's theorem (see e.g. DALEY and VERE-JONES (1988)) yields

$$P^!_{\lambda,Q\,;\,[x_{m_1},K_{m_1}],\dots,[x_{m_k},K_{m_k}],[x_{n_1},K_{n_1}],\dots,[x_{n_j},K_{n_j}]} \;=\; P_{\lambda,Q}\,, \qquad (27)$$

and for any measurable function $\nu([y,L])$ on $\mathbb{R}^d\times\mathbb{K}$ with $0\le\nu([y,L])\le 1$ for all $[y,L]\in\mathbb{R}^d\times\mathbb{K}$ the generating functional of Ψ becomes

$$\int_{N_{\mathbb{K}}}\prod_{[y,L]\,\in\,S_\psi}\nu([y,L])\,P_{\lambda,Q}(d\psi) \;=\; \exp\Big(-\lambda\int_{\mathbb{R}^d\times\mathbb{K}}\Big(1-\nu([y,L])\Big)\,dy\,Q(dL)\Big)\,. \qquad (28)$$

Using (26), (27), and (28) we obtain that

$$\begin{aligned}&\mathbb{E}\sum_{\substack{m_1,\dots,m_k\ge 1\\ n_1,\dots,n_j\ge 1}}^{*}\mathbb{1}\,(A_{q_1,\dots,q_{i-k}\,;\,m_1,\dots,m_k\,;\,n_1,\dots,n_j})\\
&= \lambda^{k+j}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\int_{\mathbb{K}}\cdots\int_{\mathbb{K}}\prod_{[y,L]\,\in\,\{[x_{q_1},K_{q_1}],\dots,[x_{q_{i-k}},K_{q_{i-k}}]\}}\mathbb{1}_{\check L}(y)\cdot\mathbb{1}_{\check L+h}(y)\\
&\qquad\prod_{[y,L]\,\in\,\{[x_{m_1},K_{m_1}],\dots,[x_{m_k},K_{m_k}]\}}\mathbb{1}_{\check L}(y)\cdot\Big(1-\mathbb{1}_{\check L+h}(y)\Big)\\
&\qquad\prod_{[y,L]\,\in\,\{[x_{n_1},K_{n_1}],\dots,[x_{n_j},K_{n_j}]\}\setminus\{[x_{q_1},K_{q_1}],\dots,[x_{q_{i-k}},K_{q_{i-k}}]\}}\Big(1-\mathbb{1}_{\check L}(y)\Big)\cdot\mathbb{1}_{\check L+h}(y)\\
&\qquad\exp\Big(-\lambda\int_{\mathbb{R}^d\times\mathbb{K}}\Big(\mathbb{1}_{\check L}(y)+\mathbb{1}_{\check L+h}(y)-\mathbb{1}_{\check L\cap(\check L+h)}(y)\Big)\,dy\,Q(dL)\Big)\\
&\qquad dx_{m_1}\cdots dx_{m_k}\cdot dx_{n_1}\cdots dx_{n_j}\cdot Q(dK_{m_1})\cdots Q(dK_{m_k})\cdot Q(dK_{n_1})\cdots Q(dK_{n_j})\\
&= \lambda^{k+j}\Big(\int_{\mathbb{K}}|\check L\cap(\check L+h)|\,Q(dL)\Big)^{i-k}\Big(\int_{\mathbb{K}}\Big(|\check L|-|\check L\cap(\check L+h)|\Big)\,Q(dL)\Big)^{k+j-(i-k)}\\
&\qquad\exp\Big(-\lambda\int_{\mathbb{K}}\Big(2\,|\check L|-|\check L\cap(\check L+h)|\Big)\,Q(dL)\Big)\\
&= \lambda^{k+j}\Big(\mathbb{E}\,|M_0\cap(M_0+h)|\Big)^{i-k}\Big(\mathbb{E}\,|M_0|-\mathbb{E}\,|M_0\cap(M_0+h)|\Big)^{j-i+2k}\\
&\qquad\exp\Big(-\lambda\Big(2\,\mathbb{E}\,|M_0|-\mathbb{E}\,|M_0\cap(M_0+h)|\Big)\Big)\,.\end{aligned}$$

This formula combined with relation (25) proves Theorem 4.1 for $1 \le i \le j$. In particular, setting $i = j \ge 1$ and $h = o$, we get a simple expression for $\mathbb{P}(o \in \Xi^{(i)})$; see Corollary 4.2 below. If $1 \le j \le i$, we make use of the fact that

$$\mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)}) = \mathbb{P}(o \in \Xi^{(j)}, -h \in \Xi^{(i)}) = \mathbb{P}(o \in \Xi^{(j)}, h \in \Xi^{(i)}) ,$$

where the first equality is due to stationarity and the second follows from the above-derived formula for our multiphase Boolean model. Finally, for the remaining cases, e.g. $i = 0$, $j \ge 1$, we rewrite $\mathbb{P}(o \in \Xi^{(0)}, h \in \Xi^{(j)})$ as

$$\mathbb{P}(o \in \Xi^{(0)}, h \in \Xi^{(j)}) = \mathbb{P}(o \in \Xi^{(j)}) - \sum_{i \ge 1} \mathbb{P}(o \in \Xi^{(i)}, h \in \Xi^{(j)}) .$$

A short computation yields that

$$\mathbb{P}(o \in \Xi^{(0)}, h \in \Xi^{(j)}) = \frac{\lambda^j \left(\mathbb{E}\,|M_0| - \mathbb{E}\,|M_0 \cap (M_0 + h)|\right)^j}{j!} \exp\left(-\lambda\, \mathbb{E}\,|M_0 \cup (M_0 + h)|\,\right)$$

completing the proof of Theorem 4.1.

As mentioned in the foregoing proof, Theorem 4.1 yields a well–known formula for the probability $\mathbb{P}(o \in \Xi^{(i)})$, which coincide with the volume fraction $\mathsf{E}|\Xi^{(i)} \cap [0,1]^d|$ of that subset of the Boolean model Ξ that is covered by exactly i of the shifted grains $M_n + X_n$, $n \ge 1$; see, for example, Chapter 4 in Hall (1988).

Corollary 4.2 *Under the assumptions of Theorem 4.1, for any $i \in \mathbb{N} \cup \{0\}$ it holds*

$$\mathbb{P}(o \in \Xi^{(i)}) \quad = \quad \frac{(\lambda\, \mathbb{E}\,|M_0|)^i}{i!} \exp(-\lambda\, \mathbb{E}\,|M_0|) .$$

Remarks

1. For the multiphase model deduced from the Boolean model in (21) with r different phases Ξ_k, $1 \le k \le r$, we are able to verify the conditions (13) and (14) of Theorem 3.1. After some rather lengthy and technical calculations it turns out that not only condition (13) but also condition (14) (or, alternatively, (16), (17) resp. (19)) is satisfied whenever the volume of the typical grain M_0 possesses a finite second moment, i.e. $\mathbb{E}\,|M_0|^2 < \infty$. This condition also implies the validity of the central limit theorem in (6) for the above–defined multiphase Boolean model, which can be seen by applying a central limit theorem for m_n–dependent random fields with $m_n \to \infty$ as $n \to \infty$ (see Heinrich (1988)) combined with the Cramér–Wold device.

2. The estimate $\int_{\mathbb{R}^d} |c_{kl}^{(1,1)}(x)|\, dx \le 5\, \lambda\, \mathbb{E}\,|M_0|^2$ holds for all $k, l \ge 1$. To prove this we use (23), (24) and the representation formulae given in Theorem 4.1 and Corollary 4.2. With the abbreviations $a(x) = \lambda\, \mathbb{E}\,|M_0 \cap (M_0 + x)|$ and $a = a(o)$ it is easily seen that $|c_{kl}^{(1,1)}(x)|$ is bounded by

$$\sum_{i,j \ge 1} \left|\, \mathbb{P}(o \in \Xi^{(i)}, x \in \Xi^{(j)}) - \mathbb{P}(o \in \Xi^{(i)})\, \mathbb{P}(x \in \Xi^{(i)}) \,\right|$$

$$\leq \sum_{i,j\geq 1} \sum_{m=1}^{\min\{i,j\}} \frac{a^m(x)\,(a-a(x))^{i+j-2m}}{m!\,(i-m)!\,(j-m)!}\, e^{a(x)-2\,a}$$
$$+\sum_{i,j\geq 1} \frac{|(a-a(x))^{i+j}-a^{i+j}|}{i!\,j!}\, e^{-2\,a} + \sum_{i,j\geq 1} \frac{(a-a(x))^{i+j}}{i!\,j!}\left(e^{a(x)-2\,a}-e^{-2\,a}\right)$$

for any $k,l \geq 1$. After evaluating the multiple sums we obtain that the right-hand side of the latter inequality does not exceed $5\,(1-e^{-\,a(x)})$. Finally, by the elementary inequality $1-e^{-\,a(x)} \leq a(x)$ and Fubini's theorem we get

$$\int_{\mathbb{R}^d} \left(1-e^{-\,a(x)}\right) dx \leq \lambda \int_{\mathbb{R}^d} \mathbb{E}\,|M_0 \cap (M_0+x)|\, dx = \lambda\, \mathbb{E}\,|M_0|^2 \tag{29}$$

proving the asserted estimate.

3. To verify condition (14) we rewrite the phase sets Ξ_k and Ξ_l as countable unions of the disjoint RACS's of the form (22). This leads to the estimate

$$\begin{aligned} &|\ \mathbb{P}(o \in \Xi_l \cap (\Xi_k - x), z \in \Xi_l \cap (\Xi_k - y)) \\ &\qquad\qquad - \mathbb{P}(o \in \Xi_l \cap (\Xi_k - x))\ \mathbb{P}(o \in \Xi_l \cap (\Xi_k - y))\ | \\ &\leq \sum_{i,j,p,q \geq 1} |\ \mathbb{P}(o \in \Xi^{(i)}, x \in \Xi^{(j)}, z \in \Xi^{(p)}, y+z \in \Xi^{(q)}) \\ &\qquad\qquad - \mathbb{P}(o \in \Xi^{(i)}, x \in \Xi^{(j)})\ \mathbb{P}(o \in \Xi^{(p)}, y \in \Xi^{(q)})\ | \end{aligned} \tag{30}$$

for any $k,l \geq 1$. Following the lines of the proof of Theorem 4.1 we arrive at a series representation for the probabilities $\mathbb{P}(o \in \Xi^{(i)}, x \in \Xi^{(j)}, z \in \Xi^{(p)}, y+z \in \Xi^{(q)})$ by infinite series in which, among others, exponential terms the form $\exp\{-\lambda\,\mathbb{E}\,|M_0 \cup (M_0-x) \cup (M_0-y) \cup (M_0-y-z))|\}$ appear. Together with the formulae for the probabilities $\mathbb{P}(o \in \Xi^{(i)}, x \in \Xi^{(j)})$ and $\mathbb{P}(o \in \Xi^{(p)}, y \in \Xi^{(q)})$ given in Theorem 4.1 it can be shown that the four–fold sum in (30) is bounded by

$$c_1\, a(z) + c_2\, a(y+z) + c_3\, a(z-x) + c_4\, a(y+z-x)\ ,$$

where $c_1,\dots,c_4$ are constants depending only on λ and $\mathbb{E}\,|M_0|$. The details of the computations are omitted here. Integrating the latter expression with respect to z over $\mathbb{R}^d$ and taking into account (29) we get that the corresponding integral over the left-hand side of (30) is bounded by $\lambda\,(c_1+\cdots+c_4)\,\mathbb{E}\,|M_0|^2$ for all $x,y \in \mathbb{R}^d$. Thus, condition (14) is satisfied if again $\mathbb{E}\,|M_0|^2 < \infty$. In a similar way and under the same condition one can check that (19) is satisfied whereas (18) requires the stronger assumption $\mathbb{E}\,|M_0|^4 < \infty$.

4. For completeness we mention that the assumption $\mathbb{E}\,|M_0|^2 < \infty$ implies that

$$\sup_{x_1,\dots,x_k \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\Gamma_{k+2}(\xi(0),\xi(x_1),\dots,\xi(x_k),\xi(z))|\, dz < \infty \quad \text{for any} \quad k \geq 1\ ,$$

where $\xi(x) = \mathbb{1}_{\Xi}(x)$ and Ξ is the Boolean model defined by (21). This estimate is a consequence of Lemma 7 in Heinrich (2004) and obtained by an inductive proving technique which also yields that $\mathbb{E}\,|M_0|^{k+1} < \infty$ is sufficient to hold

$$\int_{\mathbb{R}^{k\,d}} |\Gamma_{k+1}(\xi(0),\xi(x_1),\ldots,\xi(x_k))|\, dx_1 \ldots dx_k < \infty \quad \text{for} \quad k \geq 1\ .$$

5 Asymptotic test for volume fractions

In this section we presuppose that the multivariate central limit theorem considered in (6) holds and the asymptotic covariance matrix Σ_r has a strictly positive determinant $\det(\Sigma_r)$. Further, assume that the symmetric, non–negative definite matrix $\widehat{\Sigma}_{n,r}$ defined by (10) is a (weakly) consistent estimator for Σ_r which is guaranteed by the conditions imposed on $(\Xi_1, \ldots, \Xi_r)^\top$ in Theorem 3.1. By standard arguments one can show that $||\widehat{\Sigma}_{n,r} - \Sigma_r|| \xrightarrow[n\to\infty]{\mathbb{P}} 0$ implies

$$\det(\widehat{\Sigma}_{n,r}) \xrightarrow[n\to\infty]{\mathbb{P}} \det(\Sigma_r) > 0\,,$$

which in turn yields

$$1\!\!1\,(\det(\widehat{\Sigma}_{n,r}) > 0)\, \widehat{\Sigma}_{n,r}^{-1/2}\, \Sigma_r^{1/2} \xrightarrow[n\to\infty]{\mathbb{P}} I_r\,,$$

where I_r stands for the r–dimensional unit matrix and $\Sigma_r^{1/2}$ (resp. $\widehat{\Sigma}_{n,r}^{-1/2}$) denotes the square root of Σ_r (resp. $\widehat{\Sigma}_{n,r}^{-1}$). There are computationally efficient ways of calculating $\widehat{\Sigma}_{n,r}^{-1/2}$ based on the spectral decomposition

$$\widehat{\Sigma}_{n,r} = Q_n \operatorname{diag}(\widehat{\lambda}_{n,1}, ..., \widehat{\lambda}_{n,r})\, Q_n^\top \quad ,$$

where $\widehat{\lambda}_{n,1}, ..., \widehat{\lambda}_{n,r}$ are the positive eigenvalues of $\widehat{\Sigma}_{n,r}$ (provided that $\det(\widehat{\Sigma}_{n,r}) > 0$) and Q_n is an orthogonal $(r\times r)$–matrix whose colums are eigenvectors of $\widehat{\Sigma}_{n,r}$. Thus,

$$\widehat{\Sigma}_{n,r}^{-1/2} = Q_n \operatorname{diag}(\widehat{\lambda}_{n,1}^{-1/2}, ..., \widehat{\lambda}_{n,r}^{-1/2})\, Q_n^\top$$

is a possible choice for the square root of $\widehat{\Sigma}_{n,r}^{-1}$. There is an alternative approach, known as Cholesky decomposition, to obtain the product representation $\widehat{\Sigma}_{n,r} = \widehat{\Sigma}_{n,r}^{1/2}(\widehat{\Sigma}_{n,r}^{1/2})^\top$, where $\widehat{\Sigma}_{n,r}^{1/2}$ can be chosen as a lower triangular $(r\times r)$–matrix. For details we refer the reader to Chapter 3 in Cressie (1993).

Summarizing the above facts and using Slutsky–type arguments we are in a position to state the following result: Under the hypothesis $H_0 : (p_1, \ldots, p_r)^\top = (p_1^0, \ldots, p_r^0)^\top$ we have

$$Z_{n,r} = 1\!\!1\,(\det(\widehat{\Sigma}_{n,r}) > 0)\, \widehat{\Sigma}_{n,r}^{-1/2} \begin{pmatrix} \sqrt{|W_n|}\,(\widehat{p}_{W_n,1} - p_1^0) \\ \vdots \\ \sqrt{|W_n|}\,(\widehat{p}_{W_n,r} - p_r^0) \end{pmatrix} \overset{d}{\underset{n\to\infty}{\Longrightarrow}} N(0, I_r)\,,$$

which, by applying the continuous mapping theorem (see e.g. Daley and Vere-Jones (1988)), implies

$$||Z_{n,r}||^2 \overset{d}{\underset{n\to\infty}{\Longrightarrow}} ||N(0, I_r)||^2 \quad ,$$

where the random variable $||N(0, I_r)||^2$ is χ^2–distributed with r degrees of freedom. Thus, for a given significance level α , we will reject the hypothesis $H_0 : (p_1, \ldots, p_r)^\top = (p_1^0, \ldots, p_r^0)^\top$ if the test statistic $||Z_{n,r}||^2$ exceeds the critical value $\chi^2_{r,1-\alpha}$, which is determined by the equation $\mathbb{P}(||N(0, I_r)||^2 > \chi^2_{r,1-\alpha}) = \alpha$.

In the particular case $r = 2$ we get $\chi^2_{2,1-\alpha} = -2\ln(\alpha)$ (since $||N(0, I_2)||^2$ is exponentially distributed with mean 2) so that the hypothesis $H_0 : (p_1, p_2)^\top = (p_1^0, p_2^0)^\top$ is rejected if $||Z_{n,2}|| > \sqrt{-2\ \ln(\alpha)}$.

6 Numerical example

To illustrate the fit of the estimator $\widehat{C}_{W,kl}(h)$ given in (9) to the theoretical covariance $C_{kl}(h)$ in (4), we consider the realization of the multiphase model deduced from the Boolean model Ξ in $\mathbb{R}^2$ with two different phases Ξ_1 and Ξ_2 within a sampling window W.

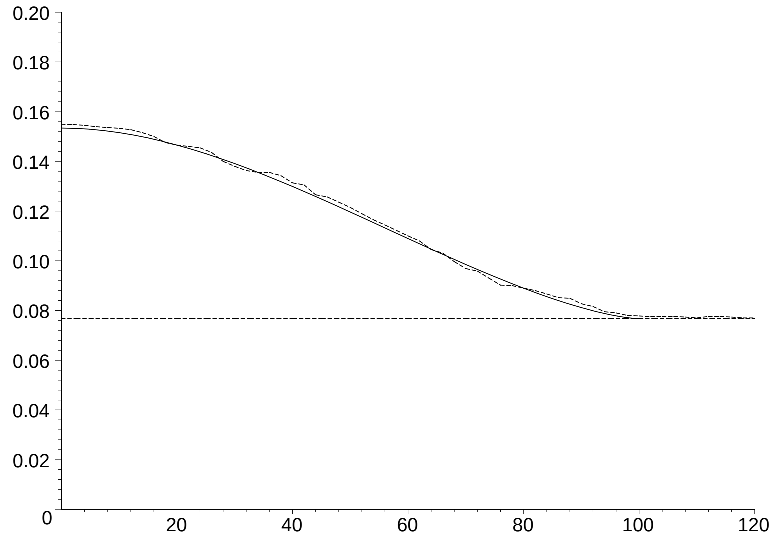

Figure 2: Comparison of $C_{12}(h)$ (——) and $\overline{C}_{W,12}(r)$ (- - -) for the multiphase model shown in Figure 1, where $p_1 = 0.5$, $p_2 = 0.1534$, and $p_1\, p_2 = 0.0767$.

If we additionally assume that Ξ is isotropic, the covariance $C_{kl}(h)$ depends only on the radial coordinate $r = ||h||$. Notice that the estimation of $C_{kl}(h)$ from a two–dimensional image can only be done for vectors h, which take values on a two–dimensional lattice. Therefore, in the isotropic case it is convenient to estimate the covariance using the *rotation average* $\overline{C}_{W,kl}(r)$, where

$$\overline{C}_{W,kl}(r) = \frac{1}{\#\{h : ||h|| \approx r\}} \sum_{h:\, ||h|| \approx r} \widehat{C}_{W,kl}(h)\,.$$

Here, the set $\{h : ||h|| \approx r\}$ can be obtained using the so–called *midpoint circle algorithm*, which is a common algorithm in image analysis to detect a circle on a lattice; see, for example, Hearn and Baker (1997). This averaging method enables us to improve the accuracy of the estimation.

An efficient way to compute the estimator $\widehat{C}_{W,kl}(h)$ from the observed image is to use methods from spectral analysis; see, for example, Böhm, Heinrich and Schmidt (2004), Ohser and Mücklich (2000), and Press, Flannery, Teukolsky and Vetterling (2002). In particular, we compare $C_{12}(h)$ using Theorem 4.1

with the corresponding estimator $\widehat{C}_{W,12}(h)$ obtained from the realization of the multiphase model shown in Figure 1. In Figure 2 it can be seen that the estimator $\overline{C}_{W,12}(r)$ fits the theoretical covariance $C_{12}(r)$ quite good for the considered range of $0 \leq r \leq 120$.

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References

BÖHM, S., L. HEINRICH and V. SCHMIDT (2004), Kernel estimation of the spectral density of stationary random closed sets. *Australian & New Zealand Journal of Statistics* **46**, 41–51.

CRESSIE, N.A.C. (1993), *Statistics for Spatial Data* (2nd ed.), J. Wiley & Sons, New York.

DALEY, D.J. and D. VERE-JONES (1988), *An Introduction to the Theory of Point Processes*, Springer, New York.

GUYON, X. (1995), *Random Fields on a Network*, Springer, New York.

HALL, P. (1988), *Introduction to the Theory of Coverage Processes*, J. Wiley & Sons, Chichester.

HEARN, D. and M.P. BAKER (1997), *Computer Graphics*, Prentice Hall, New Jersey.

HEINRICH, L. (1988), Asymptotic behaviour of an empirical nearest-neighbour distance function for stationary Poisson cluster processes, *Mathematische Nachrichten* **136**, 131–148.

HEINRICH, L. (2004), Large deviations of the empirical volume fraction for stationary Poisson grain models, *Annals of Applied Probability* **14** (to appear).

HEINRICH, L. and Z. PAWLAS (2004), Weak and strong convergence of empirical distribution functions in germ–grain models, Preprint, University of Augsburg (submitted).

IVANOV, A.V. and N.N. LEONENKO (1989), *Statistical Analysis of Random Fields*, Kluwer, Dortrecht.

KLENK, S., V. SCHMIDT and E. SPODAREV (2004), A new algorithmic approach to the computation of Minkowski functionals of polyconvex sets. Preprint. University of Ulm (submitted).

MASE, S. (1982), Asymptotic properties of stereological estimators of the volume fraction for stationary random sets, *Journal of Applied Probability* **19**, 111–126.

MOLCHANOV, I. (1997), *Statistics of the Boolean Model for Practitioners and Mathematicians*, J. Wiley & Sons, Chichester.

OHSER, J. and F. MÜCKLICH (2000), *Statistical Analysis of Microstructures in Materials Science*, J. Wiley & Sons, Chichester.

PRESS, W.H., B.P. FLANNERY, S.A. TEUKOLSKY and W.T. VETTERLING (2002), *Numerical Recipes in* C++ (2nd ed.), Cambridge University Press, Cambridge.

SCHMIDT, V. and E. SPODAREV (2004), Joint estimators for the specific intrinsic volumes of stationary random sets. *Stochastic Processes and Their Applications* (to appear)

STOYAN, D., W.S. KENDALL and J. MECKE (1995), *Stochastic Geometry and its Applications* (2nd ed.), J. Wiley & Sons, Chichester.

WILLS, J.M. (1970), Zum Verhältnis von Volumen zu Oberfläche bei konvexen Körpern, Archiv der Mathematik **21**, 557–560.