Some Properties of the EWMA Control Chart in the Presence of Autocorrelation

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Abstract

Schmid (1996) extended the classical EWMA control chart to autocorrelated processes. Here, we consider the tail probability of the run length in the in-control state. The in-control process is assumed to be a stationary Gaussian process. It is proved that the tails for the autocorrelated process are larger than in the case of independent variables if all autocovariances are greater or equal to zero. The inequality is strict. Moreover, this result is still valid for stationary processes having elliptically contoured marginal distributions.

1 Introduction

For a long time it was always assumed in statistical process control that the observations of the underlying process are independent. Unfortunately, this assumption is frequently violated in practice (Box/Jenkins/MacGregor (1974), Montgomery (1991, ch. 8–8)).

Several authors discussed how the classical Shewhart, EWMA and CUSUM control charts behave for autocorrelated processes (e.g. Alwan/Roberts (1988), Harris/Ross (1991), Woodall/Faltin (1994)). It has turned out that these schemes are not suitable if the same control limits are used as in the case of independent variables. For this reason it is necessary to apply time series models to construct control charts.

Harris/Ross (1991) and Montgomery/Mastrangelo (1991) analyzed residual charts. A classical control scheme for independent variables is applied to the residuals of the process. This procedure is permitted as long as the residuals are independent.

Another attempt was made by Vasilopoulos/Stamboulis, already in 1978. They introduced a modified Shewhart chart for autoregressive processes. The distance between the mean of the sample and the target value is compared with the standard

Key words and phrases: Statistical process control, EWMA control chart, autocorrelation, Gaussian process, average run length.

AMS 1991 subject classifications: 62N10, 62L10, 60G10.

deviation of the autocorrelated process. The run length of this chart was studied in Schmid (1995).

Here, we consider an extension of the EWMA control chart to autocorrelated processes which was proposed by Schmid (1996).

Let $\{Y_t\}$ be the in-control process. Let $x_1, x_2, \ldots$ denote realizations of the observed process $\{X_t\}$ and let $\mu_0 := E(Y_t)$ be the target value. If e.g. a shift occurs at time q, i.e.

$$X_t = Y_t + aI_{\{q,q+1,\ldots\}}(t), \tag{1}$$

only the mean of the process $\{X_t\}$ is affected.

The EWMA control scheme bases on the statistic

$$Z_t := (1-\lambda)Z_{t-1} + \lambda X_t \qquad \text{for } t \in \mathbb{N}, \lambda \in (0,1]. \tag{2}$$

The starting value $Z_0 := z_0$ is frequently taken to be equal to the target value μ_0.

It is concluded that the process is out-of-control at time t if

$$|Z_t - \mu_0| > c\sqrt{\mathrm{Var}_0\, Z_t}.$$

The symbol $\mathrm{Var}_0\, Z_t$ denotes the variance of Z_t with respect to the in-control process. For a change point model (see (1)) $\mathrm{Var}_0\, Z_t$ is equal to the variance of Z_t with respect to the process $\{X_t\}$.

Schmid (1996) compared this control scheme with the classical EWMA chart applied to the residuals. As a measure of performance the average run length (ARL) was used. $\{Y_t\}$ was assumed to be an autoregressive process of order one. For a specified value of the shift the smoothing parameter λ was chosen such that the out-of-control ARL is minimal. It has turned out that the minimum out-of-control ARL of the modified EWMA chart is smaller than that of the residual chart unless the coefficient of the autoregressive process is not too small (roughly: $\alpha \geq -0.6$).

In this paper the run length in the in-control state, i.e. for $X_t = Y_t$ for all t, of the one-sided modified EWMA chart is considered. For reasons of simplicity we write $\mathrm{Var}\, Z_t$ instead of $\mathrm{Var}_0\, Z_t$. The run length is given by

$$N_e := \inf\left\{t \in \mathbb{N} \,:\, Z_t - \mu_0 > c\sqrt{\mathrm{Var}\, Z_t}\right\}, \tag{3}$$

where $c > 0$ denotes a given constant. $\{Y_t\}$ is assumed to be a stationary Gaussian process. Our main result (Theorem 1) states that the run length of the autocorrelated process is larger than in the case of independent variables provided that all autocovariances are greater or equal to zero.

2 Main Result

In the following we use the notation P_{iid}, E_{iid}, Var_{iid} and Corr_{iid} to refer to the case of independent variables $\{Y_t\}$ with $E(Y_t) = \mu_0$ and $\text{Var}\, Y_t = \gamma_0$ for all t. Corr stands for the correlation. Now, let $C_k := (\text{Cov}\,(Y_s, Y_t))_{1 \le s,t \le k}$.

Theorem 1: *Assume that $\{Y_t\}$ is a stationary Gaussian process with autocovariance function $\{\gamma_\nu\}$ satisfying $\gamma_\nu \ge 0$ for all ν and $\gamma_0 > 0$. If furthermore $z_0 = \mu_0$, then*

$$P(N_e > k) \ge P_{\text{iid}}(N_e > k) \tag{4}$$

for $k \in \mathbb{N}_0$. The inequality is strict for $k \ge 2$ if $\gamma_\nu > 0$ for at least one $\nu \in \{1, \ldots, k-1\}$.

The proof is given in section 3.

Theorem 1 states that the tails of the stopping rule N_e are larger if the process is autocorrelated. This implies that $E(N_e^\kappa) \ge E_{\text{iid}}\ (N_e^\kappa)$ for $\kappa \in \mathbb{N}_0$. Consequently the ARL of the modified EWMA control chart for autocorrelated variables is always greater or equal to that in the case of independent variables. Thus, if one falsely assumes that the underlying process is independent, the true ARL is underestimated.

Example: Let $\{Y_t\}$ be the stationary solution of $Y_t = \alpha Y_{t-1} + \varepsilon_t, \quad \alpha \in [0, 1)$. Assume that the random variables $\{\varepsilon_t\}$ are independent and identically distributed with $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$ (normal distribution with mean 0 and variance σ^2). Thus $\{Y_t\}$ is a Gaussian process and $\gamma_\nu = \sigma^2 \alpha^{|\nu|} / (1 - \alpha^2) \ge 0$ for all ν. Since $P_{\text{iid}}(N_e > k)$ is equal to $P(N_e > k)$ if α is taken to be zero Theorem 1 implies that $P_\alpha(N_e > k) \ge P_0(N_e > k)$. Here the index expresses that the probability is taken with respect to an AR(1)-process with coefficient α.

Remarks:

(a) If $\lambda = 1$ and $\alpha \in (-1, 0)$, then $P(N_e > 2) = P(Y_1/\sqrt{\gamma_0} \le c,\, Y_2/\sqrt{\gamma_0} \le c) < P_{\text{iid}}(N_e > 2)$ (e.g. Tong (1980), Theorem 2.1.1). This shows that Theorem 1 is no longer valid if the autocovariances may be negative.

(b) Theorem 1 remains true if N_e is replaced by the stopping rule $\inf\{t \in \mathbb{N} \mid Z_t - \mu_0 < c\sqrt{\text{Var}\, Z_t}\}$.

(c) Using a generalization of Slepian's inequality for elliptically contoured distributions (see Tong (1980), Theorem 4.3.6) it is possible to extend Theorem 1 to stochastic processes whose marginal distributions are elliptically contoured.

3 Proof of Theorem 1

The proof of Theorem 1 is based on the following three lemmas.

Lemma 1: *Let $\{Y_t\}$ be a (weakly) stationary process with mean μ_0 and autocovariance function $\{\gamma_h\}$. Furthermore let $\lambda \in (0,1)$. Then we obtain for $i, t, t-i \in I\!N_0$ that*

$$2\,Cov(Z_t, Z_{t-i}) = (1-\lambda)^i \; Var\, Z_{t-i} + \frac{1}{(1-\lambda)^i}\left(Var Z_t - \; Var\, Z_i\right).$$

Proof: The recursion (2) can be written as follows

$$Z_t = (1-\lambda)^i Z_{t-i} + \lambda \sum_{\nu=0}^{i-1} (1-\lambda)^\nu Y_{t-\nu}.$$

It follows that

$$Z_t - E(Z_t) = (1-\lambda)^i \left(Z_{t-i} - E(Z_{t-i})\right) + \lambda \sum_{\nu=0}^{i-1} (1-\lambda)^\nu \left(Y_{t-\nu} - \mu_0\right). \tag{5}$$

Multiplying (5) on both sides by $Z_{t-i} - E(Z_{t-i})$ and taking the expectation yields

$$\operatorname{Cov}(Z_t, Z_{t-i}) = (1-\lambda)^i \operatorname{Var} Z_{t-i} + \lambda \sum_{\nu=0}^{i-1} (1-\lambda)^\nu \operatorname{Cov}(Y_{t-\nu}, Z_{t-i}). \tag{6}$$

Also, multiplying (5) on both sides by $Z_t - E(Z_t)$ and taking the expectation, we get

$$\begin{aligned}
\operatorname{Var} Z_t &= (1-\lambda)^i \operatorname{Cov}(Z_t, Z_{t-i}) + \lambda \sum_{\nu=0}^{i-1} (1-\lambda)^\nu \operatorname{Cov}(Y_{t-\nu}, Z_t) \\
&= (1-\lambda)^i \operatorname{Cov}(Z_t, Z_{t-i}) + \lambda \sum_{\nu=0}^{i-1} (1-\lambda)^{\nu+i} \operatorname{Cov}(Y_{t-\nu}, Z_{t-i}) \\
&\qquad + \lambda^2 \sum_{\nu,\mu=0}^{i-1} (1-\lambda)^{\nu+\mu} \gamma_{\nu-\mu}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\lambda \sum_{\nu=0}^{i-1} (1-\lambda)^\nu \operatorname{Cov}(Y_{t-\nu}, Z_{t-i}) \\
&\quad = \frac{\operatorname{Var} Z_t}{(1-\lambda)^i} - \operatorname{Cov}(Z_t, Z_{t-i}) - \lambda^2 \sum_{\nu,\mu=0}^{i-1} (1-\lambda)^{\nu+\mu-i} \gamma_{\nu-\mu} \\
&\quad = \frac{\operatorname{Var} Z_t}{(1-\lambda)^i} - \operatorname{Cov}(Z_t, Z_{t-i}) - \frac{\operatorname{Var} Z_i}{(1-\lambda)^i},
\end{aligned} \tag{7}$$

since

$$\operatorname{Var} Z_i = \lambda^2 \sum_{\nu,\mu=0}^{i-1} (1-\lambda)^{\nu+\mu} \gamma_{\nu-\mu}. \tag{8}$$

Inserting (7) in (6) the result follows. □

Next we compare the behaviour of two successive variances. Note that equation (8) implies that $\operatorname{Var} Z_i \geq \lambda^2 \gamma_0$ since $\gamma_\nu \geq 0$ for $\nu \geq 1$. Assuming $\gamma_0 > 0$ we obtain

$\mathrm{Var}\, Z_i \geq \lambda^2\, \mathrm{Var}_{\mathrm{iid}} Z_i > 0$.

Lemma 2: *Assume that $\{Y_t\}$ satisfies the assumptions of Lemma 1. Moreover, let $\gamma_\nu \geq 0$ for all ν and $\gamma_0 > 0$. Then, for $t \geq 1$*

$$\frac{Var\, Z_{t-1}}{Var\, Z_t} \leq \frac{Var_{\mathrm{iid}} Z_{t-1}}{Var_{\mathrm{iid}} Z_t}. \tag{9}$$

For $t \geq 2$ the inequalitiy is strict if $\gamma_\nu > 0$ for at least one $\nu \in \{1, \ldots, t-1\}$.

Proof: Let $x := 1 - \lambda$ and $\rho_\nu := \gamma_\nu / \gamma_0$. Because of (8) the inequality (9) is equivalent to

$$\left(x^{2(t-1)} - x^{2t}\right) \sum_{\nu,\mu=0}^{t-1} x^{\nu+\mu} \rho_{\nu-\mu} \leq \left(1 - x^{2t}\right) \left(\sum_{\nu,\mu=0}^{t-1} x^{\nu+\mu} \rho_{\nu-\mu} - \sum_{\nu,\mu=0}^{t-2} x^{\nu+\mu} \rho_{\nu-\mu} \right).$$

Since

$$\sum_{\nu,\mu=0}^{t-1} x^{\nu+\mu} \rho_{\nu-\mu} - \sum_{\nu,\mu=0}^{t-2} x^{\nu+\mu} \rho_{\nu-\mu} = 2x^{t-1} \sum_{\mu=0}^{t-2} x^{\mu} \rho_{t-1-\mu} + x^{2(t-1)},$$

we get

$$\sum_{\substack{\nu,\mu=0 \\ \nu \neq \mu}}^{t-1} x^{\nu+\mu} \rho_{\nu-\mu} \leq 2x^{1-t} \frac{1 - x^{2t}}{1 - x^2} \sum_{\mu=0}^{t-2} x^{\mu} \rho_{t-1-\mu}. \tag{10}$$

Now, using induction, it can easily be proved that (10) is true. For $t = 2$ the proof is obvious. Let us only consider the induction step ($t \rightsquigarrow t+1$)

$$\begin{aligned} \sum_{\substack{\nu,\mu=0 \\ \nu \neq \mu}}^{t} x^{\nu+\mu} \rho_{\nu-\mu} &\leq 2x^t \sum_{\mu=0}^{t-1} x^{\mu} \rho_{t-\mu} + 2x^{-t} \frac{1 - x^{2t}}{1 - x^2} x \sum_{\mu=0}^{t-2} x^{\mu} \rho_{t-1-\mu} \\ &\leq 2x^{-t} \frac{1 - x^{2(t+1)}}{1 - x^2} \sum_{\mu=0}^{t-1} x^{\mu} \rho_{t-\mu}. \end{aligned}$$

The above inequality is strict if $\gamma_\nu > 0$ for at least one $\nu \in \{1, \ldots, t\}$. □

Lemma 2 implies that

$$\frac{\mathrm{Var}\, Z_{t-i}}{\mathrm{Var}\, Z_t} = \prod_{\nu=0}^{i-1} \frac{\mathrm{Var}\, Z_{t-\nu-1}}{\mathrm{Var}\, Z_{t-\nu}} \leq \frac{\mathrm{Var}_{\mathrm{iid}} Z_{t-i}}{\mathrm{Var}_{\mathrm{iid}} Z_t}. \tag{11}$$

The next result plays a central role in the proof of Theorem 1. Its proof is based on the preceeding lemmas.

Lemma 3: *Assume that the conditions of Lemma 2 are satisfied. Then, for i, t, $t - i \in \mathbb{N}_0$*

$$Corr(Z_t, Z_{t-i}) \geq Corr_{\mathrm{iid}}(Z_t, Z_{t-i}).$$

For $i, t, t-i \in \mathbb{N}$ the inequality is strict if $\gamma_\nu > 0$ for at least one $\nu \in \{1, \ldots, t-1\}$.

Proof: Because of Lemma 1 it follows that

$$\begin{aligned}
2\,\mathrm{Corr}(Z_t, Z_{t-i}) &= (1-\lambda)^i \sqrt{\frac{\mathrm{Var}\,Z_{t-i}}{\mathrm{Var}\,Z_t}} + \frac{1}{(1-\lambda)^i}\sqrt{\frac{\mathrm{Var}\,Z_t}{\mathrm{Var}\,Z_{t-i}}}\left(1 - \frac{\mathrm{Var}\,Z_i}{\mathrm{Var}\,Z_t}\right) \\
&\geq (1-\lambda)^i \sqrt{\frac{\mathrm{Var}\,Z_{t-i}}{\mathrm{Var}\,Z_t}} + \frac{1}{(1-\lambda)^i}\sqrt{\frac{\mathrm{Var}\,Z_t}{\mathrm{Var}\,Z_{t-i}}}\left(1 - \frac{\mathrm{Var}_{\mathrm{iid}} Z_i}{\mathrm{Var}_{\mathrm{iid}} Z_t}\right) \qquad (12) \\
&= (1-\lambda)^i \left(\sqrt{\frac{\mathrm{Var}\,Z_{t-i}}{\mathrm{Var}\,Z_t}} + \sqrt{\frac{\mathrm{Var}\,Z_t}{\mathrm{Var}\,Z_{t-i}}}\,\frac{\mathrm{Var}_{\mathrm{iid}} Z_{t-i}}{\mathrm{Var}_{\mathrm{iid}} Z_t}\right), \qquad (13)
\end{aligned}$$

since by (8)

$$\frac{\mathrm{Var}_{\mathrm{iid}} Z_{t-i}}{\mathrm{Var}_{\mathrm{iid}} Z_t} = \frac{1}{(1-\lambda)^{2i}}\left(1 - \frac{\mathrm{Var}_{\mathrm{iid}} Z_i}{\mathrm{Var}_{\mathrm{iid}} Z_t}\right).$$

The right side of (13) is of the type $g(z) := a(z + b/z)$ which is nonincreasing for $z^2 \leq b$. Thus, we get by (11)

$$g\left(\sqrt{\frac{\mathrm{Var}\,Z_{t-i}}{\mathrm{Var}\,Z_t}}\right) \geq g\left(\sqrt{\frac{\mathrm{Var}_{\mathrm{iid}} Z_{t-i}}{\mathrm{Var}_{\mathrm{iid}} Z_t}}\right) = 2\,\mathrm{Corr}_{\mathrm{iid}}\,(Z_t, Z_{t-i})$$

and the result follows.

According to Lemma 2 the inequality (12) is strict if $\gamma_\nu > 0$ for at least one $\nu \in \{1, \ldots, t-1\}$. □

Now we are able to give a proof of our main result.

Proof of Theorem 1: First note that

$$P(N_e > k) = P\left(\max_{1\leq i\leq k} \frac{Z_i - \mu_0}{\sqrt{\mathrm{Var}\,Z_i}} \leq c\right).$$

The random vector $\left((Z_1 - \mu_0)/\sqrt{\mathrm{Var}\,Z_1}, \ldots, (Z_k - \mu_0)/\sqrt{\mathrm{Var}\,Z_k}\right)$ has a multivariate normal distribution with mean 0. For $\lambda = 1$ the result follows directly from Slepian's inequality (see Theorem 2.1.1 of Tong (1980)). If $\lambda \in (0, 1)$ we make use of Lemma 3 and Slepian's inequality to finish the proof. □

Acknowledgment: We would like to thank the referee for helpful comments.

References

ALWAN, L.C.; ROBERTS, H.V. (1988). Time series modeling for statistical process control. *J. Business and Economic Statist.* **6**, 87–95.

BOX, G.E.P.; JENKINS, G.M.; MACGREGOR, J.F. (1974). Some recent advances in forecasting and control, Part II. *J. Roy. Statist. Soc. Ser. C* **23**, 158–179.

HARRIS, T.J.; ROSS, W.H. (1991). Statistical process control procedures for correlated observations. *Canadian J. Chemical Engineering* **69**, 48–57.

MONTGOMERY, D.C. (1991). *Statistical Quality Control.* Wiley, New York.

MONTGOMERY, D.C.; MASTRANGELO, C.M. (1991). Some statistical process control methods for autocorrelated data. *J. Quality Technology* **23(3)**, 170–193.

SCHMID, W. (1995). On the run length of a Shewhart chart for correlated data. *Statistical Papers* **36**, *111-130.*

SCHMID, W. (1996). On EWMA charts for time series. To appear in *Frontiers of Statistical Quality Control.*

TONG, Y.L. (1980). *Probability Inequalities in Multivariate Distributions.* Academic Press, New York.

VASILOPOULOS, A.V.; STAMBOULIS, A.P. (1978). Modification of control chart limits in the presence of data correlation. *J. Quality Technology* **10(1)**, 20–30.

WOODALL, W.H.; FALTIN, F.W. (1994). Autocorrelated data and SPC. *ASQC Statist. Divison Newsletter* **13(4)**, 18–21.

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