The Design of a Fuzzy - Shewhart Control Chart

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1 Introduction

In this paper we suggest a design for a control chart in the case of vague data. Such data may arise in form of linguistic data or by imprecise measurements, which all can modelled by fuzzy - sets. We consider fuzzy - sets as real valued interpretations of vague data. We leave the traditional approach that we are always able to assign exactly one real number (ore p-dimensional vectors) to the outcome of our experiment. Indeed, if we have to meet uncertainty and vagueness in describing our observations, we should replace the classical dichotomous judgements (we observe "x", $x \in \mathbb{R}$) by introducing fuzzy - sets, which themselves can be described by membership functions. For quality purposes, such an approach is described in more detail e.g. in Laviolette et al. [14].

Our starting point is based on the pioneering work of L.A.Zadeh [26, 27] on fuzzy - sets and his extension principle as well as on the concept of a fuzzy random variable introduced by Kwakernaak [11]. The generalization of traditional control charts to the "fuzzy case" will be studied in this paper for the simple Shewhart chart. The statistics involved in Shewhart chart techniques are the mean and the variance of a random sample. Therefore, we need a generalization of such statistics for fuzzy random samples. The corresponding statistics can be found e.g. in the excellent book of R.Kruse and K.D.Meyer [10], which gives a thorough discussion of statistical methods for fuzzy - data.

Even for Shewhart charts, we will see that in case of fuzzy - data the stopping rule will not be as simple as in the traditional case. Simultaneous test problems arise implying e.g. difficulties for an exact evaluation of the operating characteristic function or the average run length.

2 Fuzzy - sets and some properties

In this section, we want to describe the concept of fuzzy - sets based on the work of L.A.Zadeh [26, 27]. We give the most important tools of fuzzy - sets needed for the construction of the fuzzy - Shewhart control chart.

Definition 2.1 :

(i) A fuzzy - set of the real line $\mathbb{R}$ is characterized by its membership function μ , where :

$\mu \, : \, \mathbb{R} \longrightarrow [0,1]\,.$

We denote by $E(\mathbb{R})$ the class of all fuzzy - sets of $\mathbb{R}$.

(ii) A fuzzy - set $\mu \in E(\mathbb{R})$ of the real line is called **normal** , if

$\exists\, t \in \mathbb{R} \quad : \quad \mu(t) = 1$.

We denote by $F(\mathbb{R})$ the class of all normal fuzzy - sets of $\mathbb{R}$.

Example 2.2 :

(i) The characteristic function yields a simple fuzzy - set.
Let $A \subseteq \mathbb{R}$ be a non-empty subset of $\mathbb{R}$. Then it holds :

$$I_A(t) := \begin{cases} 1 & , \text{if } \ t \in A \\ 0 & , \text{if } \ t \in \mathbb{R} \backslash A \end{cases}$$

is a normal fuzzy - set.

(ii) $\mu(t) := \max\{1 - (t-2)^2, 0\}$ is a normal fuzzy - set.

Important tools for dealing with fuzzy - sets are their α-level-sets and their strong α-cuts, which were discussed by C.V.Negoita and D.A.Ralescu [19].

Definition 2.3 :

Let $\mu \in F(\mathbb{R})$, $\alpha \in [0,1]$.

Then define :

(i) $\mu_{\overline{\alpha}} := \{t \in \mathbb{R} \,|\, \mu(t) \geq \alpha\}$ as **α-level-set.**

(ii) $\mu_{\alpha} := \{t \in \mathbb{R} \,|\, \mu(t) > \alpha\}$ as **strong α-cut.**

M.Miakoshi and M.Shimbo [15] discussed the concept of set representations of fuzzy - sets. This leads to a generalization of the α-level-set and the strong α-cut defined by C.V.Negoita and D.A.Ralescu.

Definition 2.4 :

Let $\mu \in F(\mathbb{R})$.

$\{A_\alpha \,|\, \alpha \in (0,1)\}$ is called a **set representation of μ** if and only if

(i) $0 \leq \alpha \leq \beta < 1 \quad \Longrightarrow \quad A_\beta \subseteq A_\alpha \subseteq \mathbb{R}$.

(ii) For all $t \in \mathbb{R} : \mu(t) = \sup\{\alpha I_{A_\alpha}(t) \,|\, \alpha \in (0,1)\}$, where $I_{A_\alpha}(t)$ denotes the characteristic function of A_α.

The strong α-cut respectively the α-level-set are the "smallest", respectively the "biggest", set representation in the sense of M.Miakoshi and M.Shimbo. We remark that $\{\mu_{\overline{\alpha}} \,|\, \alpha \in (0,1)\}$ and $\{\mu_\alpha \,|\, \alpha \in (0,1)\}$ are both set representations of the fuzzy - set μ.
Necessary and sufficient conditions that $\{A_\alpha \,|\, \alpha \in (0,1)\}$ is a set representation of a fuzzy - set μ are given in the following theorem (M.Miyakoshi and M.Shimbo [15]), which is very useful for proofs.

Theorem 2.5 :

Let $\mu \in F(\mathbb{R})$ and $\{A_\alpha \,|\, \alpha \in (0,1)\}$ be a system of subsets of $\mathbb{R}$. Then it holds :

$\{A_\alpha \,|\, \alpha \in (0,1)\}$ is a set representation of μ if and only if

$\mu_\alpha \subseteq A_\alpha \subseteq \mu_{\overline{\alpha}}$ holds for all $\alpha \in (0,1)$.

Now, we give an example of a fuzzy - set and the way how to get the set representation for a given fuzzy - set.

Example 2.6 :

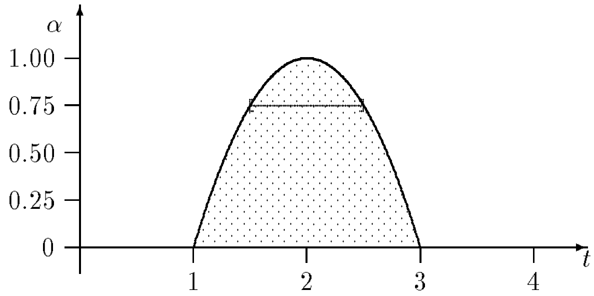
Let $\mu(t) := \max\{1 - (t-2)^2, 0\}$ be a fuzzy - set. Then, we have :

$$\begin{aligned} \mu_\alpha &= \{t \in \mathbb{R} \,|\, \mu(t) > \alpha\} \\ &= \{t \in \mathbb{R} \,|\, 2 - \sqrt{1-\alpha} < t < 2 + \sqrt{1-\alpha}\} \\ \Longrightarrow \quad \mu_\alpha &= (2 - \sqrt{1-\alpha}, 2 < + \sqrt{1-\alpha}). \end{aligned}$$

in the same way we get :

$\mu_{\overline{\alpha}} = [2 - \sqrt{1-\alpha}, 2 + \sqrt{1-\alpha}]$.

The corresponding graphical representation is given below :



The extension priciple of L.A.Zadeh offers a powerful tool for generalizing classical mathematical concepts to the fuzzy case.

Definition 2.7 :(Extension principle of Zadeh, see R.R.Yager [25])

Let $n \in \mathbb{N}$, $(\mu_1 \ \ldots, \mu_n) \in [F(\mathbb{R})]^n$, $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a map.
Then, we call the fuzzy number $\Phi[\mu_1, \ldots, \mu_n]$ with

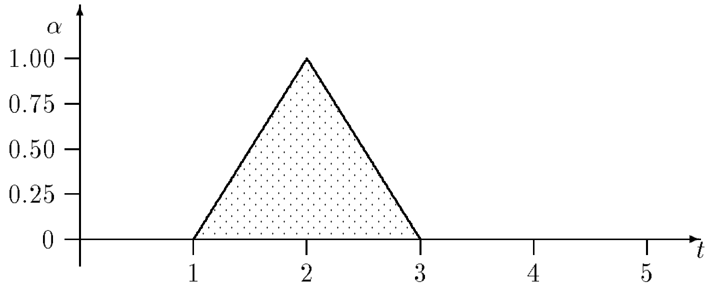
$$\Phi[\mu_1, \ldots, \mu_n](t) := \sup \left\{ \min\{\mu_1(t_1), \ldots, \mu_n(t_n)\} \;\middle|\; \begin{array}{l} (t_1, \ldots, t_n) \in \mathbb{R}^n, \\ \Phi(t_1, \ldots, t_n) = t \end{array} \right\}$$

for all $t \in \mathbb{R}$ the **image of** $(\boldsymbol{\mu_1, \ldots, \mu_n})$ **under** Φ**.**

By using the extension principle of Zadeh, we are able to extend the usual mathematical operations between exact data to operations between fuzzy - sets. The following example shows how the extension principle works.

Example 2.8 :

$$\mu(t) = \begin{cases} t-1 & , \text{if} \quad 1 \le t \le 2 \\ 3-t & , \text{if} \quad 2 < t \le 3 \\ 0 & , \text{otherwise.} \end{cases}$$



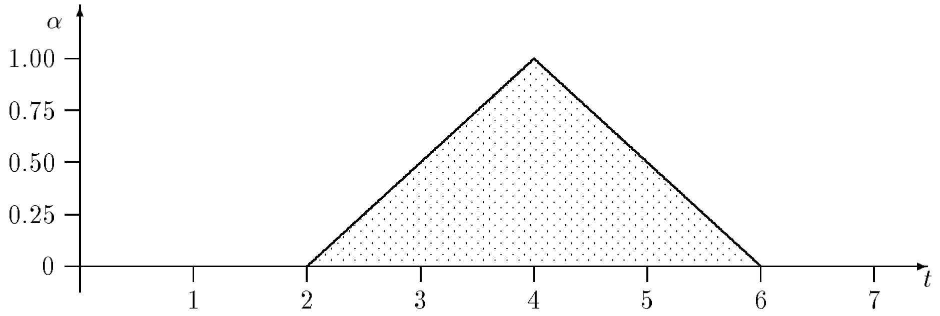
Let $\Phi(t) := 2\,t$.

$$\Longrightarrow \quad \Phi[\mu](t) = \sup\{\mu(\mathrm{v}) \mid \mathrm{v} \in \mathbb{R},\ 2\mathrm{v} = \mathrm{t}\}$$

$$= \sup\{\mu(\mathrm{v}) \mid \mathrm{v} \in \mathbb{R},\ \mathrm{v} = \tfrac{\mathrm{t}}{2}\}$$

$$= \sup\{\mu(\tfrac{t}{2}) \mid t \in \mathbb{R}\}$$

$$= \mu(\tfrac{t}{2})\,.$$



The addition of fuzzy - sets and the multiplication of a fuzzy - set by a scalar can be done easier by using only strong α-cuts or α-level-sets, respectively.

Definition 2.9 :

Let $A,\, B \subseteq \mathbb{R}$ be non-empty sets and $\lambda \in \mathbb{R}$.

Then, the **Minkowski** operations are defined by

(i) $A + B := \{x + y \mid x \in A,\, y \in B\}$,

(ii) $\lambda \cdot A := \{\lambda \cdot x \mid x \in A\}$.

For the design of the fuzzy - Shewhart control chart we want to work with a smaller class of fuzzy - sets than the class of normal fuzzy - sets $F(\mathbb{R})$. This leads to the class of convex fuzzy - sets. The motivation for dealing with this smaller class is e.g. also given by the importance of this class for application purposes.

Definition 2.10 :

(i) The class of all **fuzzy - sets with a normal set representation** is denoted by $Q(\mathbb{R})$. It holds $\mu \in Q(\mathbb{R})$ if and only if there is a set representation $\{A_\alpha \,|\, \alpha \in (0,1)\}$ of μ such that

(a) $\inf A_\alpha > -\infty$ implies $\inf A_\alpha \in A_\alpha$.

(b) $\sup A_\alpha < \infty$ implies $\sup A_\alpha \in A_\alpha$.

(c) If $\inf A_\alpha = -\infty$ or $\sup A_\alpha = \infty$ then for all $\alpha \in (0,1)$ A_α is convex.

(ii) The class of all **convex fuzzy - sets** is denoted by $U(\mathbb{R})$.
It holds $\mu \in U(\mathbb{R})$ if and only if

(a) $\mu \in F(\mathbb{R})$,

(b) there exist a set representation $\{A_\alpha \,|\, \alpha \in (0,1)\}$ of μ with A_α is convex and closed for all $\alpha \in (0,1)$.

(iii) The class of all **bounded convex fuzzy - sets** is denoted by $U_c(\mathbb{R})$.
It holds $\mu \in U_c(\mathbb{R})$ if and only if

(a) $\mu \in U(\mathbb{R})$,

(b) $\inf \mu_0 > -\infty$ and $\sup \mu_0 < +\infty$.

The class of convex fuzzy - sets has nice properties, which makes the work much easier (see e.g. R.Kruse and K.D.Meyer [10]).

Theorem 2.11 :

Let $n \in \mathbb{N}$, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\alpha \in (0,1)$.

(i) If $(\mu_1, \ldots, \mu_n) \in [U(\mathbb{R})]^n$, then :

(a) $\sum\limits_{i=1}^{n} \lambda_i \mu_i \in U(\mathbb{R})$.

(b) $\left(\sum\limits_{i=1}^{n} \lambda_i \mu_i\right)_\alpha = \sum\limits_{i=1}^{n} \lambda_i (\mu_i)_\alpha$.

(ii) If $(\mu_1, \ldots, \mu_n) \in [U_c(\mathbb{R})]^n$, then :

(a) $\sum\limits_{i=1}^{n} \lambda_i \, \mu_i \in U_c(\mathbb{R})$.

(b) $\left(\sum\limits_{i=1}^{n} \lambda_i \, \mu_i \right)_{\overline{\alpha}} = \sum\limits_{i=1}^{n} \lambda_i \, (\mu_i)_{\overline{\alpha}}$.

3 The fuzzy random variable

The theory of fuzzy random variables is based on two different types of uncertainty, namely vagueness and randomness. Fuzzy random variables were discussed e.g. by H.Kwakernaak [11].

Let (Ω, Σ, P) be a probability space. On this probability space we will define a random mechanism. For many applications, we have the situation that our randomness will be aditionally disturbed by errors of measurement, genetic differences, etc. These effects are not fuzzy effects. Therefore, we need a second probability space (Ω', Σ', P') in order to describe these external effects. For the probability space (Ω', Σ', P'), we assume :

For all $\lambda \in [0,1]$ there exist $A' \in \Sigma'$ with $P'(A') = \lambda$.

The random mechanism of the fuzzy random variable depends on the product space of both probability spaces (compare e.g. E.Henze [5]), separating the random mechanism from the above described disturbances. The probability space is defined by :

$(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) = (\Omega \times \Omega', \Sigma \otimes \Sigma', P \otimes P')$.

Now, the whole random mechanism under consideration can be be described by an usual random variable on the probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$. Hence, an interpretation of a fuzzy random variable may be given as follows :
"A fuzzy random variable (f.r.v.) is a (fuzzy) perception of an unknown, usual random variable."

We call the possible random variables of this random experiment the originals of the fuzzy random variable. The vagueness of the observation can be described by a fuzzy - set. The fuzzy random variable depends on the first probability space (Ω, Σ, P) and is independent of the second one (Ω', Σ', P'), on the other hand we can't observe the disturbance directly. The possible originals of the f.r.v. are dependend of both probability spaces and therefore of the product space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$.

Now, we are led to the following definition of a f.r.v. (compare e.g. H.Kwakernaak [11]) :

Definition 3.1 :

A mapping $X : \Omega \longrightarrow Q(\mathbb{R})$ is called a fuzzy random variable (f.r.v.), if there is a system $\{ A_\alpha(\omega) | \omega \in \Omega, \alpha \in (0,1) \}$ of subsets of $\mathbb{R}$ with the following properties :

(i) $\{ A_\alpha(\omega) | \alpha \in (0,1) \}$ is a normal set representation of $X_\omega := X(\omega)$ for all $\omega \in \Omega$.

(ii) The mappings $\underline{A}_\alpha : \Omega \to \mathbb{R}$ and $\overline{A}_\alpha : \Omega \to \mathbb{R}$ are Σ-$\mathcal{B}_1$-measurable for all $\alpha \in (0,1)$, where $\underline{A}_\alpha(\omega) := \inf A_\alpha(\omega)$ and $\overline{A}_\alpha(\omega) := \sup A_\alpha(\omega)$.

For the following analysis, we have to make a few assumptions :

Assumption 3.2 :

(i) We restrict the definition of a f.r.v. to the class of convex fuzzy sets, i.e. a f.r.v. X is a map

$X : \Omega \longrightarrow U(\mathbb{R})$ respectively $U_c(\mathbb{R})$.

(ii) The possible originals with respect to the random experiment are unknown, but we assume that the originals U are normally distributed :

$U \sim N(\mu_0, \sigma_0^2)$, $\mu_0 \in \mathbb{R}$, $\sigma_0^2 > 0$.

Therefore, the set $\tilde{\chi}$ of all possible originals is given by all usual normally distributed random variables on $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$.

With regard to assumption (i) it has been shown e.g. R.Kruse and K.D.Meyer [10] that we will get the same results using either the set $\tilde{\chi}$ or χ. The set χ is given by all normally distributed random variables on (Ω, Σ, P). Therefore, we restrict the following considerations to the easier case, where $X : \Omega \longrightarrow U(\mathbb{R})$ is a f.r.v., and $U : \Omega \longrightarrow \mathbb{R}$ is a possible original from the set χ. Then, by definition of the fuzzy logic the acceptability of the vague statement "U is original of X" can be calculated by :

$\mu_X(U) = \inf\{ X_\omega(U(\omega)) | \omega \in \Omega \}$.

4 The expectation of a fuzzy random variable

For the design of a control chart within the framework of quality control with fuzzy data, we do need the expectation and the variance of a f.r.v. R.Kruse and

K.D.Meyer [10] developed a concept for calculating these quantities. Again, this concept is based on the extension principle of Zadeh. The expectation and the variance of a f.r.v. are fuzzy - sets.

Definition 4.1 :

Let (Ω, Σ, P) be a probability space, $X : \Omega \longrightarrow U(\mathbb{R})$ be a f.r.v. and $t \in \mathbb{R}$:

(i) $E : \chi \longrightarrow U(\mathbb{R})$ with

$$(E\,X)(t) := \sup \left\{ \inf_{\omega\in\Omega} \mu_{X(\omega)}(U(\omega)) \;\middle|\; \begin{array}{l} U \in \chi, \\ E\,|U^k| < \infty\,, \\ E\,U^k = t \end{array} \right\}$$

is called the **expectation of X with respect to** χ

(ii) $\mathrm{Var} : \chi \longrightarrow U(\mathbb{R})$ with

$$(\mathrm{Var}\,X)(t) := \sup \left\{ \inf_{\omega\in\Omega} \mu_{X(\omega)}(U(\omega)) \;\middle|\; \begin{array}{l} U \in \chi, \\ \mathrm{Var}\,|U| < \infty\,, \\ \mathrm{Var}\,U = t \end{array} \right\}$$

is called the **variance of X with respect to** χ **.**

These definitions seem to be too complicated for calculations. But set representation leads to a feasible method for computing the expectation.

Theorem 4.2 :

Let $X : \Omega \longrightarrow U(\mathbb{R})$ be a f.r.v. such that $E\,|\underline{Y}_0| < \infty$, $E\,|\overline{Y}_0| < \infty$, and define for $\omega \in \Omega$ and $\alpha \in [0,1)$:

$(\underline{Y})_\alpha(\omega) := \inf(X(\omega))_\alpha\,,\ (\overline{Y})_\alpha(\omega) := \sup(X(\omega))_\alpha$

Then it holds :

$\{[E\,\underline{Y}_\alpha\,,\,E\,\overline{Y}_\alpha]\,|\,\alpha \in (0,1)\}$ is a set representation of $E\,X$.

The following remark gives us a possibility for computing the expectation of any linear combinations of f.r.v.'s.

Remark 4.3 :

Let $\lambda \in \mathbb{R}$, $X_1, X_2, X : \Omega \longrightarrow U(\mathbb{R})$ be f.r.v.'s , then it holds :

$$\begin{aligned} E(X_1 + X_2) &= E(X_1) + E(X_2), \\ E(\lambda X) &= \lambda E X. \end{aligned}$$

5 The statistics $\bar{X}$ and S

In this section, we want to discuss two of the most important statistics of a fuzzy random sample. Both are basic tools for constructing the fuzzy - Shewhart control chart.

Definition 5.1 :

Let $(X_1, \ldots, X_n) : \Omega \longrightarrow [U(\mathbb{R})]^n$.
$(X_1, \ldots, X_n)$ is called a **fuzzy random sample** if the fuzzy sample variables X_i are i.i.d. for $i \in \{1, \ldots, n\}$.

Definition 5.2 :

(i) Let $(X_1, \ldots, X_m)$ be a fuzzy random sample

(a) $\overline{X} := \frac{1}{m} \sum_{j=1}^{m} X_j$,

(b) $\overline{S}^2 := \frac{1}{m-1} \sum_{j=1}^{m} (X_j - \overline{X})^2$.

(ii) Let $(X_{i1}, \ldots, X_{im})$ be a fuzzy random sample for $i \in \{1, \ldots, n\}$

(a) $\overline{\overline{X}} := \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij}$,

(b) $\overline{S_i}^2 := \frac{1}{m-1} \sum_{j=1}^{m} (X_{ij} - \overline{X}_i)^2$,

(c) $\overline{S} := \frac{1}{n} \sum_{i=1}^{n} \overline{S}_i$.

Again, these statistics are fuzzy - sets. Since we want to use these statistics as estimators for the moments of the f.r.v.'s, we have to discuss these statistics. In the following remark, we first discribe the corresponding α-level-sets.

Remark 5.3 :

Let $\alpha \in (0,1)$ and $\overline{X}\,,\, \overline{S}^2\,,\, \overline{\overline{X}}\,,\, \overline{S_i}^2\,,\, \overline{S}$ defined as in (5.2). Then it holds :

$$\text{(i)}\quad \overline{X}_\alpha = \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\,(t_1, \ldots, t_m) \in (X_1)_\alpha \times \ldots \times (X_m)_\alpha \\ \text{such that } \frac{1}{m}\sum\limits_{j=1}^{m} t_j = t \end{array} \right\},$$

$$\text{(ii)}\quad \overline{S}^2_\alpha = \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\,(t_1, \ldots, t_m) \in (X_1)_\alpha \times \ldots \times (X_m)_\alpha \\ \text{such that } \frac{1}{m-1}\sum\limits_{j=1}^{m} (t_j - \overline{t}\,)^2 = t \end{array} \right\},$$

$$\text{(iii)}\quad \overline{\overline{X}}_\alpha = \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\,(t_{11}, \ldots, t_{nm}) \in (X_{11})_\alpha \times \ldots \times (X_{nm})_\alpha \\ \text{such that } \frac{1}{mn}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} t_{ij} = t \end{array} \right\},$$

$$\text{(iv)}\quad (\overline{S_i}^2)_\alpha = \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\,(t_{i1}, \ldots, t_{im}) \in (X_{i1})_\alpha \times \ldots \times (X_{im})_\alpha \\ \text{such that } \frac{1}{m-1}\sum\limits_{j=1}^{m} (t_{ij} - \overline{t}_i\,)^2 = t \end{array} \right\},$$

$$\text{(v)}\quad \overline{S}_\alpha = \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists\,(t_1, \ldots, t_n) \in (S_1)_\alpha \times \ldots \times (S_n)_\alpha \\ \text{such that } \frac{1}{n}\sum\limits_{i=1}^{n} t_i = t \end{array} \right\}.$$

Proof :

Applying the extension principle of Zadeh, at once, we will get these results.

For the following analysis, we do need the infimum and the supremum of these fuzzy - sets.

Lemma 5.4 :

Let $\overline{X}$ und $\overline{S}$ be defined as (5.2). Then it holds :

$$\text{(i)}\quad \inf \overline{X}_\alpha = \frac{1}{m}\sum_{j=1}^{m} \inf(X_j)_\alpha \quad , \quad \sup \overline{X}_\alpha = \frac{1}{m}\sum_{j=1}^{m} \sup(X_j)_\alpha\,,$$

$$\text{(ii)}\quad \inf \overline{S}_\alpha = \frac{1}{n}\sum_{i=1}^{n} \inf(S_i)_\alpha \quad , \quad \sup \overline{S}_\alpha = \frac{1}{n}\sum_{i=1}^{n} \sup(S_i)_\alpha\,.$$

Proof :

Again, the extension principle of Zadeh and Theorem 2.11 implies :

$$\begin{aligned}
\inf \overline{X}_\alpha &= \inf \left(\tfrac{1}{m} \sum_{j=1}^{m} X_j \right)_\alpha \\
&= \inf \tfrac{1}{m} \sum_{j=1}^{m} (X_j)_\alpha \\
&= \tfrac{1}{m} \sum_{j=1}^{m} \inf (X_j)_\alpha \qquad \square
\end{aligned}$$

The other assertions can be shown analogously.

Next, we define some entities, which will be needed for determing the control limits (case of unknown variance).

Definition 5.5 :

Let $(X_{i1}, \ldots, X_{im}) \in [U(\mathbb{R})]^m$, $i \in \{1, \ldots, n\}$, and $\overline{S}_i^2$ as in Definition 5.2 (ii) .

Define :

(i) $((\overline{S}_i^2)_{\inf})_\alpha \;:=\; \frac{1}{m-1} \sum_{j=1}^{m} (\inf (X_{ij})_\alpha - \inf (\overline{X}_i)_\alpha)^2$,

(ii) $((\overline{S}_i^2)_{\sup})_\alpha \;:=\; \frac{1}{m-1} \sum_{j=1}^{m} (\sup (X_{ij})_\alpha - \sup (\overline{X}_i)_\alpha)^2$,

(iii) $((\overline{S})_{\inf})_\alpha \;:=\; \frac{1}{n} \sum_{i=1}^{n} ((\overline{S}_i)_{\inf})_\alpha$.

Let us consider the usual random variables $\inf(\overline{X})_\alpha$, $\sup(\overline{X})_\alpha$, $((\overline{S}_i^2)_{\inf})_\alpha$, and $((\overline{S}_i^2)_{\sup})_\alpha$.

Lemma 5.6 :

Let $(X_1, \ldots, X_m) : \Omega \longrightarrow [U(\mathbb{R})]^m$ be an i.i.d. sequence of f.r.v. with :
$E\,X_j = \mu$, $\operatorname{Var} X_j = \sigma^2$ forall $j \in \{1, \ldots, m\}$, where $\mu, \sigma^2 \in U_c(\mathbb{R})$.
Define for $j \in \{1, \ldots, m\}$, $\omega \in \Omega$, $\alpha \in [0,1)$:

$(\underline{Y_j})_\alpha(\omega) := \inf((X_j)_\omega)_\alpha, \ (\overline{Y_j})_\alpha(\omega) := \sup((X_j)_\omega)_\alpha$

Then it holds :

$$\begin{array}{rlcll}
\text{(i)} & E(\inf(\overline{X})_\alpha) & = & E(\underline{Y}_1)_\alpha & = \inf \mu_\alpha, \\
\text{(ii)} & E(\sup(\overline{X})_\alpha) & = & E(\overline{Y}_1)_\alpha & = \sup \mu_\alpha, \\
\text{(iii)} & \mathrm{Var}(\inf(\overline{X})_\alpha) & = & \frac{1}{m}\mathrm{Var}(\underline{Y}_1)_\alpha, & \\
\text{(iv)} & \mathrm{Var}(\sup(\overline{X})_\alpha) & = & \frac{1}{m}\mathrm{Var}(\overline{Y}_1)_\alpha. &
\end{array}$$

Remark 5.7 :

Let us remark :

$$\begin{array}{rlcl}
\text{(i)} & \mathrm{Var}(\underline{Y}_1)_\alpha & \geq & \inf \sigma_\alpha^2, \\
\text{(ii)} & \mathrm{Var}(\overline{Y}_1)_\alpha & \leq & \sup \sigma_\alpha^2,
\end{array}$$

where in general, equality does not hold (see R.Kruse and K.D.Meyer [10]) .

Remark 5.8 :

In Lemma 5.6, we are allowed to replace $\overline{X}$ and m by $\overline{\overline{X}}$ and nm, respectively.

Lemma 5.9 :

Let $(X_{11}, \ldots, X_{nm}) : \Omega \longrightarrow [U(\mathbb{R})]^{nm}$ be a i.i.d. - sequence of f.r.v.'s with $E X_{ij} = \mu$, $\mathrm{Var}\, X_{ij} = \sigma^2$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ where $\mu, \sigma^2 \in U_c(\mathbb{R})$.
Define for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, $\omega \in \Omega$, $\alpha \in [0,1)$:
$(\underline{Y_{ij}})_\alpha(\omega) := \inf((X_{ij})_\omega)_\alpha, \ (\overline{Y_{ij}})_\alpha(\omega) := \sup((X_{ij})_\omega)_\alpha$, and let
$\gamma_m := \sqrt{\frac{2}{m-1}} \cdot \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m-1}{2})}$.

Then it holds :

$$\begin{aligned}
&\text{(i)} \quad E\,(\,(\overline{S}_i^2)_{\text{inf}}\,)_\alpha &&= \operatorname{Var}\,(\,\underline{Y}_{11}\,)_\alpha\,, \\
&\text{(ii)} \quad E\,(\,(\overline{S}_i^2)_{\text{sup}}\,)_\alpha &&= \operatorname{Var}\,(\,\overline{Y}_{11}\,)_\alpha\,, \\
&\text{(iii)} \quad E\,(\,\overline{S}_{\text{inf}}\,)_\alpha &&= \gamma_m \cdot \sqrt{\operatorname{Var}\,(\,\underline{Y}_{11}\,)_\alpha}\,, \\
&\text{(iv)} \quad E\,(\,\overline{S}_{\text{sup}}\,)_\alpha &&= \gamma_m \cdot \sqrt{\operatorname{Var}\,(\,\overline{Y}_{11}\,)_\alpha}\,.
\end{aligned}$$

Proof :

$$X_{11}\,, \cdots, X_{1m}\,, \cdots, X_{n1}\,, \cdots, X_{nm} \quad \text{i.i.d.} \quad \text{f.r.v.}$$

$$\begin{aligned}
\Longrightarrow \quad & \underline{Y}_{11}\,, \cdots, \underline{Y}_{1m}\,, \cdots, \underline{Y}_{n1}\,, \cdots, \underline{Y}_{nm} \quad \text{i.i.d.} \\
& \overline{Y}_{11}\,, \cdots, \overline{Y}_{1m}\,, \cdots, \overline{Y}_{n1}\,, \cdots, \overline{Y}_{nm} \quad \text{i.i.d.}
\end{aligned}$$

$$\begin{aligned}
\text{(i)} \quad E\,(\,(\,(\overline{S}_i^2)_{\text{inf}}\,)_\alpha\,) &= E\,(\,\tfrac{1}{m-1} \sum_{j=1}^{m} (\,\inf\,(X_{ij})_\alpha - \inf\,(\overline{X_i})_\alpha\,)^2\,) \\
&= E\,(\,\tfrac{1}{m-1} \sum_{j=1}^{m} (\,(\,\underline{Y}_{ij}\,)_\alpha \;-\; \tfrac{1}{m} \sum_{j=1}^{m} (\,\underline{Y}_{ij}\,)_\alpha\,)^2\,) \\
&= \operatorname{Var}\,(\,\underline{Y}_{11}\,)_\alpha \qquad \square
\end{aligned}$$

(ii) analogous.

$$\begin{aligned}
\text{(iii)} \quad E\,(\,\overline{S}_{\text{inf}}\,)_\alpha &= E\,(\tfrac{1}{n} \sum_{i=1}^{n} (\,(\,S_i\,)_{\text{inf}}\,)_\alpha\,) \\
&= \tfrac{1}{n} \sum_{i=1}^{n} E\,(\,(\,S_i\,)_{\text{inf}}\,)_\alpha \\
&= \tfrac{1}{n} \sum_{i=1}^{n} \gamma_m \cdot \sqrt{\operatorname{Var}\,(\,\overline{Y}_{11}\,)_\alpha} \\
&= \gamma_m \cdot \sqrt{\operatorname{Var}\,(\,\overline{Y}_{11}\,)_\alpha} \qquad \square
\end{aligned}$$

(iv) analogous.

Lemma 5.9 implies the following unbiased estimator of the standard deviation of the usual random variables $\underline{Y}_{11}\,, \cdots, \underline{Y}_{nm}$.

Remark 5.10 :

The statistics

(i) $\overline{\overline{S}} \quad := \quad \dfrac{\overline{S}}{\gamma_m}$,

(ii) $\overline{\overline{S}}_{\text{inf}} \quad := \quad \dfrac{\overline{S}_{\text{inf}}}{\gamma_m}$,

(iii) $\overline{\overline{S}}_{\text{sup}} \quad := \quad \dfrac{\overline{S}_{\text{sup}}}{\gamma_m}$.

are unbiased estimators of the standard deviation.

Now, all relevant statistics for the design of a fuzzy - Shewhart control chart are provided. First, we want to give the control limits. The question arises how to choose the control limits. In the classical situation, assuming normally distributed random variables U_i , $i \in \{1, \ldots, n\}$.

We obtain the upper and lower control limits UCL and LCL, respectively :

$$\text{UCL} : \mathbb{R}^2 \longrightarrow \mathbb{R} \text{ with } \text{UCL}\,(EU, \sqrt{\text{Var}\,U}\,) = EU + \frac{z_{1-\frac{\delta}{2}}}{\sqrt{m}} \sqrt{\text{Var}\,U}$$

and

$$\text{LCL} : \mathbb{R}^2 \longrightarrow \mathbb{R} \text{ with } \text{LCL}\,(EU, \sqrt{\text{Var}\,U}\,) = EU - \frac{z_{1-\frac{\delta}{2}}}{\sqrt{m}} \sqrt{\text{Var}\,U} ,$$

where $z_{1-\frac{\delta}{2}}$ denotes the $(1-\frac{\delta}{2})$- quantile of the standard normal distribution. Now these usual variables should be extended to fuzzy variables by applying the extension principle of Zadeh.

Theorem 5.11 :

Let LCL$(.,.)$ and UCL$(.,.)$ are control limits of the usual Shewhart chart,

$$\begin{array}{llll} \text{LCL}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (t_1, t_2) & \longrightarrow & t_1 - \frac{z_{1-\delta_1}}{\sqrt{m}} t_2 , \\ \text{UCL}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (t_1, t_2) & \longrightarrow & t_1 + \frac{z_{1-\delta_1}}{\sqrt{m}} t_2 , \end{array}$$

$\delta_1, \delta_2 \in (0,1)$ such that $\delta_1 + \delta_2 = \delta$. Let $m \in \mathbb{N}$ be the sample size and define for $(\mu_1, \mu_2) \in [F(\mathbb{R})]^2$, $\alpha \in [0,1)$, $t \in \mathbb{R}$:

$$\mathrm{LCL}_\alpha [\mu_1 , \mu_2] \; := \; \inf \left\{ u \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists \, (t_1 , t_2) \in (\mu_1)_\alpha \times (\mu_2)_\alpha \\ \text{with } \mathrm{LCL}\,(t_1 , t_2) \le u \end{array} \right\}$$

$$\mathrm{UCL}_\alpha [\mu_1 , \mu_2] \; := \; \sup \left\{ u \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists \, (t_1 , t_2) \in (\mu_1)_\alpha \times (\mu_2)_\alpha \\ \text{with } \mathrm{UCL}\,(t_1 , t_2) \ge u \end{array} \right\}$$

$E_\alpha [\mu_1 , \mu_2] :=$

$$\begin{cases} [\,\mathrm{LCL}_\alpha [\mu_1 , \mu_2] , \mathrm{UCL}_\alpha [\mu_1 , \mu_2]\,] & , \text{if } \mathrm{LCL}_\alpha [\mu_1 , \mu_2] > -\infty , \\ & \phantom{, \text{if }} \mathrm{UCL}_\alpha [\mu_1 , \mu_2] < +\infty , \\ (-\infty , \mathrm{UCL}_\alpha [\mu_1 , \mu_2]\,] & , \text{if } \mathrm{LCL}_\alpha [\mu_1 , \mu_2] = -\infty , \\ & \phantom{, \text{if }} \mathrm{UCL}_\alpha [\mu_1 , \mu_2] < +\infty , \\ [\,\mathrm{LCL}_\alpha [\mu_1 , \mu_2] , +\infty\,) & , \text{if } \mathrm{LCL}_\alpha [\mu_1 , \mu_2] > -\infty , \\ & \phantom{, \text{if }} \mathrm{UCL}_\alpha [\mu_1 , \mu_2] = +\infty , \\ \mathbb{R} & , \text{if } \mathrm{LCL}_\alpha [\mu_1 , \mu_2] = -\infty , \\ & \phantom{, \text{if }} \mathrm{UCL}_\alpha [\mu_1 , \mu_2] = +\infty , \end{cases}$$

$K [\mu_1 , \mu_2] (t) := \sup \{ \, \alpha I_{E_\alpha [\mu_1 , \mu_2]} (t) \,|\, \alpha \in (0,1) \, \}$. Then it holds :

$K : [\, F(\mathbb{R}) \,]^2 \longrightarrow U(\mathbb{R})$ is a convex fuzzy - set and $\{ \, E_\alpha [\mu_1 , \mu_2] \,|\, \alpha \in (0,1) \, \}$ is a set representation of $K [\mu_1 , \mu_2]$.

Proof :

1. Let $(\mu_1 , \mu_2) \in [\, F(\mathbb{R}) \,]^2$.
   Obviously, we get for $0 \le \alpha \le \beta < 1$:
   $$\begin{aligned} \mathrm{LCL}_\alpha [\mu_1 , \mu_2] \; & \le \; \mathrm{LCL}_\beta [\mu_1 , \mu_2] \\ & \le \; \mathrm{UCL}_\beta [\mu_1 , \mu_2] \\ & \le \; \mathrm{UCL}_\alpha [\mu_1 , \mu_2] . \end{aligned}$$

   Because the intersection of a non-increasing sequence of closed non-empty sets is not empty, i.e. $\bigcap\limits_{\alpha \in (0,1)} E_\alpha [\mu_1 , \mu_2] \ne \emptyset$, it holds :

$K[\mu_1, \mu_2] \in F(\mathbb{R})$.

Taking into account that $\{ E_\alpha[\mu_1, \mu_2] \mid \alpha \in (0,1) \}$ is a set representation of $K[\mu_1, \mu_2]$, it follows :

$K[\mu_1, \mu_2] \in U(\mathbb{R})$.

2. Let $(\mu_1, \mu_2) \in [F(\mathbb{R})]^2$, $\alpha \in [0,1)$.

   We want to show that

$$
\begin{aligned}
\text{(I)} \quad \inf(K[\mu_1, \mu_2])_\alpha &= \mathrm{LCL}_\alpha[\mu_1, \mu_2], \\
\text{(II)} \quad \sup(K[\mu_1, \mu_2])_\alpha &= \mathrm{UCL}_\alpha[\mu_1, \mu_2].
\end{aligned}
$$

   Let $\alpha_r := \alpha + \frac{1}{2r}(1 - \alpha)$.

   Then it holds :

$$
\{ \mathrm{LCL}_{\alpha_r}[\mu_1, \mu_2] \}_{r \in \mathbb{N}} \searrow \inf(K[\mu_1, \mu_2])_\alpha,
$$

$$
\{ \mathrm{UCL}_{\alpha_r}[\mu_1, \mu_2] \}_{r \in \mathbb{N}} \nearrow \sup(K[\mu_1, \mu_2])_\alpha.
$$

   (I) Let $\psi_\alpha := \mathrm{LCL}_\alpha[\mu_1, \mu_2] \le \mathrm{LCL}_{\alpha_r}[\mu_1, \mu_2]$, $r \in \mathbb{N}$.

   Choose $R \in \mathbb{N}$ and $(t_1^{(R)}, t_2^{(R)}) \in (\mu_1)_\alpha \times (\mu_2)_\alpha$ such that

$$
\begin{aligned}
\mathrm{LCL}(t_1^{(R)}, t_2^{(R)}) &\le -R & &, \text{ if } \psi_\alpha = -\infty, \\
\mathrm{LCL}(t_1^{(R)}, t_2^{(R)}) &\le \psi_\alpha + \tfrac{1}{R} & &, \text{ if } \psi_\alpha > -\infty.
\end{aligned}
$$

   Because it exists an integer $s = s(R)$ with $(t_1^{(R)}, t_2^{(R)}) \in (\mu_1)_{\alpha_s} \times (\mu_2)_{\alpha_s}$, it holds :

$$
\begin{aligned}
\lim_{r \longrightarrow \infty} \mathrm{LCL}_{\alpha_r}[\mu_1, \mu_2] &\le \mathrm{LCL}_{\alpha_s}[\mu_1, \mu_2] \\
&\le \mathrm{LCL}(t_1^{(R)}, t_2^{(R)}) \\
\Longrightarrow \quad \lim_{r \longrightarrow \infty} \mathrm{LCL}_{\alpha_r}[\mu_1, \mu_2] &\le \psi_\alpha \\
\Longrightarrow \quad \psi_\alpha &= \lim_{r \longrightarrow \infty} \mathrm{LCL}_{\alpha_r}[\mu_1, \mu_2] \\
&= \inf(K[\mu_1, \mu_2])_\alpha \qquad \square
\end{aligned}
$$

   (II) analogous.

Remark 5.12 :

If we restrict ourselves to the case $\mu_1\,,\ \mu_2\ \in\ U_c(\,\mathbb{R}\,)$, we are led to $E_\alpha\,[\mu_1\,,\,\mu_2]\ =\ [\,\mathrm{LCL}_\alpha\,[\mu_1\,,\,\mu_2]\,,\,\mathrm{UCL}_\alpha\,[\mu_1\,,\,\mu_2]\,]$.

By these results we are able to derive control limits for the fuzzy - Shewhart control chart. We assume that the expectation $\mu_0\ \in\ U_c(\,\mathbb{R}\,)$ and the variance $\sigma_0^2\ \in\ U_c(\,\mathbb{R}\,)$ are known.

Lemma 5.13 :

Let $\mu_0\,,\,\sigma_o^2\ \in\ U_c(\,\mathbb{R}\,),\ X\ :\ \Omega\ \longrightarrow\ U(\,\mathbb{R}\,)$ be a f.r.v. with normally distributed originals and $E\,X\ =\ \mu_0\ ,\ \mathrm{Var}\,X\ =\ \sigma_0^2\,.$
Let m be the sample size, $\alpha\ \in\ [0,1)\ ,\ \delta\ \in\ (0,1)\ ;\ z_{1-\delta}$ may denote the $(1-\delta)$-quantile of the standard normal distribution.
Furthermore, choose $\delta_1\,,\,\delta_2\ \in\ (0,1)$ such that $\delta_1\,+\,\delta_2\,=\,\delta\,.$

Then it holds :

$$\begin{aligned}\mathrm{LCL}_\alpha\,[\mu_0\,,\,\sigma_0] &= \inf\,(\,\mu_0\,)_\alpha\ -\ \tfrac{z_{1-\delta_1}}{\sqrt{m}}\ \sup(\sigma_0)_\alpha\,,\\ \mathrm{UCL}_\alpha\,[\mu_0\,,\,\sigma_0] &= \sup(\,\mu_0\,)_\alpha\ +\ \tfrac{z_{1-\delta_2}}{\sqrt{m}}\ \sup(\sigma_0)_\alpha\,.\end{aligned}$$

Proof :

Choose $u\in\mathbb{R}$ such that $t_1\in(\mu_0)_\alpha$ and $t_2\in(\sigma_0)_\alpha$ implies
$\mathrm{LCL}\,(t_1\,,\,t_2)\ =\ t_1\ -\ \frac{z_{1-\delta_2}}{\sqrt{m}}\,t_2\ \le\ u\,.$

Obviously, it holds :

$$t_1\ \ge\ \inf\,(\mu_0)_\alpha\,,\quad t_2\ \le\ \sup(\mu_0)_\alpha\,,$$

and therefore :

$$\inf\,(\mu_0)_\alpha\ -\ \tfrac{z_{1-\delta_2}}{\sqrt{m}}\ \sup(\mu_0)_\alpha\ \le\ \mathrm{LCL}\,(t_1\,,\,t_2)\ \le\ u$$

$$\Longrightarrow\quad \mathrm{LCL}_\alpha\,[\mu_0\,,\,\sigma_0]\ \ge\ \inf\,(\mu_0)_\alpha\ -\ \tfrac{z_{1-\delta_2}}{\sqrt{m}}\ \sup(\mu_0)_\alpha\,.$$

Similarly, we show

$$\mathrm{UCL}_\alpha\,[\mu_0\,,\,\sigma_0]\ \le\ \sup(\mu_0)_\alpha\ +\ \tfrac{z_{1-\delta_2}}{\sqrt{m}}\ \sup(\mu_0)_\alpha \qquad \square$$

If there is at least one of the parameters μ_0 and σ_0 unknown, as in the classical situation, then we can get corresponding fuzzy - Shewhart control charts by replacing these parameters by their estimates, which are discussed in section 4. The different cases are treated in the following Lemma.

Lemma 5.14 :

The control limits for the fuzzy - Shewhart control chart are given by :

(i) μ_0 and σ_o^2 known, $\mu_0\,,\,\sigma_o^2 \in U_c(\mathbb{R})$:

$$\begin{aligned}
\mathrm{LCL}_\alpha[\mu_0\,,\,\sigma_0] &= \inf(\mu_0)_\alpha - \tfrac{z_{1-\delta_1}}{\sqrt{m}}\sup(\sigma_0)_\alpha\,,\\
\mathrm{UCL}_\alpha[\mu_0\,,\,\sigma_0] &= \sup(\mu_0)_\alpha + \tfrac{z_{1-\delta_2}}{\sqrt{m}}\sup(\sigma_0)_\alpha\,,
\end{aligned}$$

(ii) μ_0 known, σ_o^2 unknown, $\mu_0\,,\,\overline{\overline{S}} \in U_c(\mathbb{R})$:

$$\begin{aligned}
\mathrm{LCL}_\alpha[\mu_0\,,\,\overline{\overline{S}}] &= \inf(\mu_0)_\alpha - \tfrac{z_{1-\delta_1}}{\sqrt{m}}\sup(\overline{\overline{S}})_\alpha\,,\\
\mathrm{UCL}_\alpha[\mu_0\,,\,\overline{\overline{S}}] &= \sup(\mu_0)_\alpha + \tfrac{z_{1-\delta_2}}{\sqrt{m}}\sup(\overline{\overline{S}})_\alpha\,,
\end{aligned}$$

(iii) μ_0 unknown, σ_o^2 known, $\overline{\overline{X}}\,,\,\sigma_o^2 \in U_c(\mathbb{R})$:

$$\begin{aligned}
\mathrm{LCL}_\alpha[\overline{\overline{X}}\,,\,\sigma_0] &= \inf(\overline{\overline{X}})_\alpha - \tfrac{z_{1-\delta_1}}{\sqrt{m}}\sup(\sigma_0)_\alpha\,,\\
\mathrm{UCL}_\alpha[\overline{\overline{X}}\,,\,\sigma_0] &= \sup(\overline{\overline{X}})_\alpha + \tfrac{z_{1-\delta_2}}{\sqrt{m}}\sup(\sigma_0)_\alpha\,,
\end{aligned}$$

(iv) μ_0 and σ_o^2 unknown, $\overline{\overline{X}}\,,\,\overline{\overline{S}} \in U_c(\mathbb{R})$:

$$\begin{aligned}
\mathrm{LCL}_\alpha[\overline{\overline{X}}\,,\,\overline{\overline{S}}] &= \inf(\overline{\overline{X}})_\alpha - \tfrac{z_{1-\delta_1}}{\sqrt{m}}\sup(\overline{\overline{S}})_\alpha\,,\\
\mathrm{UCL}_\alpha[\overline{\overline{X}}\,,\,\overline{\overline{S}}] &= \sup(\overline{\overline{X}})_\alpha + \tfrac{z_{1-\delta_2}}{\sqrt{m}}\sup(\overline{\overline{S}})_\alpha\,.
\end{aligned}$$

6 The fuzzy - Shewhart control chart

In the last section, we discussed the general method of computing control limits for fuzzy - Shewhart control charts. For application purposes, two questions remain to be solved.

1. The stopping rule of the fuzzy - Shewhart - control chart.

2. The choice of the parameters $\delta_1, \delta_2 \in (0,1)$.

The problem which arises is that in principle we have to analyse infinite many α - levels ($\alpha \in (0,1)$) of the corresponding statistic. Following R.Kruse and K.D.Meyer [10], we suggest the following general procedure.

1. Choose an integer N as the number of inspection α-levels.
2. Choose $K \in \{1, \ldots, N\}$ arbitrary.
3. Choose the set of α-levels $\{\alpha_1, \ldots, \alpha_N\} \subseteq [0,1)$ arbitrary.
4. Reject the null hypothesis, if at least K out of N α-levels are rejected.

This procedure, applied to the fuzzy control chart, leads to the following stopping rule.

Stopping rule for the fuzzy - Shewhart control chart

Stop the process if and only if in at least K out of N α-levels it holds :
$\mathrm{cl}(\overline{X})_{\alpha_i} \not\subseteq [\mathrm{LCL}_{\alpha_i}[\mu_1, \mu_2], \mathrm{UCL}_{\alpha_i}[\mu_1, \mu_2]]$, where

$N \in \mathbb{N}$, $K \in \{1, \ldots, N\}$, $\{\alpha_1, \ldots, \alpha_N\} \subseteq [0,1)$, $i \in \{1, \ldots, N\}$, $\mu_1, \mu_2 \in U_c(\mathbb{R})$.

Next we want to find optimal values for $\delta_1, \delta_2 \in (0,1)$ for the case "μ_0, σ_0 known". This will be done by studying the operating characteristic function.
The following Lemma shows, how to calculate the operating characteristic function for every α-level.

Lemma 6.1 :

Let $\alpha \in [0,1)$, $\mu_0 \in U_c(\mathbb{R})$ and $\sigma_0^2 \in U_c(\mathbb{R})$ be the given expectation and variance of the f.r.v. X, and let $X_1, \ldots, X_m$ be a fuzzy - random sample.

Hence, for every $\alpha \in [0,1)$ it holds :

$$\begin{aligned} L^\alpha(\mu) &:= P(\{\omega \in \Omega \,|\, \mathrm{cl}(\overline{X}_\omega)_\alpha \subseteq [\mathrm{UEG}_\alpha[\mu_0, \sigma_0], \mathrm{OEG}_\alpha[\mu_0, \sigma_0]]\}) \\ &\geq \Phi\left(\frac{\sup(\mu_0)_\alpha - \sup(\mu)_\alpha}{\sqrt{\mathrm{Var}\,\overline{Y}_\alpha}} \cdot \sqrt{m} + z_{1-\delta_2}\right) - \\ &\quad \Phi\left(\frac{\inf(\mu_0)_\alpha - \inf(\mu)_\alpha}{\sqrt{\mathrm{Var}\,\underline{Y}_\alpha}} \cdot \sqrt{m} - z_{1-\delta_1}\right), \end{aligned}$$

where $(\underline{Y})_\alpha(\omega) := \inf(X(\omega))_\alpha$, $(\overline{Y})_\alpha(\omega) := \sup(X(\omega))_\alpha$.

Proof :

Let $N \in \mathbb{N}$, $X_1, \ldots, X_m$ be a fuzzy random sample and $\overline{X} = \frac{1}{m} \sum_{j=1}^{m} X_j$.

For abbreviation, define :

$\psi_\alpha := \inf (\overline{X})_\alpha$ and $\xi_\alpha := \sup (\overline{X})_\alpha$.

Choose $(U_1^{(N)}, \ldots, U_m^{(N)}) \in \chi_\alpha^m$ and $(V_1^{(N)}, \ldots, V_m^{(N)}) \in \chi_\alpha^m$ such that

$$
\begin{array}{lcll}
\overline{U}^{(N)}(\omega) & \leq & \psi_\alpha + \frac{1}{N} & , \text{ if } \psi_\alpha > -\infty \,, \\
\overline{U}^{(N)}(\omega) & \leq & -N & , \text{ if } \psi_\alpha = -\infty \,, \\
\overline{V}^{(N)}(\omega) & \geq & \xi_\alpha - \frac{1}{N} & , \text{ if } \xi_\alpha < \infty \,, \\
\overline{V}^{(N)}(\omega) & \geq & N & , \text{ if } \xi_\alpha = \infty \,.
\end{array}
$$

Furthermore, for the originals let us assume :

$$
\begin{array}{lclcccl}
E\, U_1^{(N)} & = & \nu^{(N)}, & \nu^{(N)} & \in & E\, X_\alpha & = \mu_\alpha \,, \\
\text{Var } U_1^{(N)} & = & \tau^{2\,(N)}, & \tau^{2\,(N)} & \in & \text{Var } X_\alpha & = (\sigma_0^2)_\alpha \,, \\
E\, V_1^{(N)} & = & \lambda^{(N)}, & \lambda^{(N)} & \in & E\, X_\alpha & = \mu_\alpha \,, \\
\text{Var } V_1^{(N)} & = & \vartheta^{2\,(N)}, & \vartheta^{2\,(N)} & \in & \text{Var } X_\alpha & = (\sigma_0^2)_\alpha \,.
\end{array}
$$

Under the assumptions of our model $\nu_0^{(N)} \in (\mu_0)_\alpha$ and $\lambda_0^{(N)} \in (\mu_0)_\alpha$ exist, where $\nu_0^{(N)}$ and $\lambda_0^{(N)}$ are the expectations of the random variables $U_j^{(N)}$ and $V_j^{(N)}$, $j \in \{1, \ldots, m\}$, in the in-control-state, respectively. It holds:

$$
P\left(\left\{ \omega \in \Omega \;\middle|\; \overline{U}^{(N)}(\omega) \;\geq\; \nu_0^{(N)} - \frac{z_{1-\delta_1}}{\sqrt{m}}\, \tau_0^{(N)} \right\} \right)
$$

$$
= \; P\left(\left\{ \omega \in \Omega \;\middle|\; \frac{\overline{U}^{(N)}(\omega) - \nu^{(N)}}{\tau_0^{(N)}} \sqrt{m} \;\geq\; \frac{\nu_0^{(N)} - \nu^{(N)}}{\tau_0^{(N)}} \sqrt{m} - z_{1-\delta_1} \right\} \right)
$$

$$
= \; 1 - \Phi\left(\frac{\nu_0^{(N)} - \nu^{(N)}}{\tau_0^{(N)}} \sqrt{m} - z_{1-\delta_1} \right)
$$

and

$$P\left(\left\{\omega \in \Omega \;\middle|\; \overline{V}^{(N)}(\omega) \;\le\; \lambda_0^{(N)} + \frac{Z_{1-\delta_2}}{\sqrt{m}}\,\vartheta_0^{(N)}\right\}\right)$$

$$= \; P\left(\left\{\omega \in \Omega \;\middle|\; \frac{\overline{V}^{(N)}(\omega) - \lambda^{(N)}}{\vartheta_0^{(N)}}\sqrt{m} \;\le\; \frac{\lambda_0^{(N)} - \lambda^{(N)}}{\vartheta_0^{(N)}}\sqrt{m} - z_{1-\delta_2}\right\}\right)$$

$$= \; \Phi\left(\frac{\lambda_0^{(N)} - \lambda^{(N)}}{\vartheta_0^{(N)}}\sqrt{m} - z_{1-\delta_2}\right).$$

Define for $N \in \mathbb{N}$:

$$A_N \;:=\; \left\{\omega \in \Omega \;\middle|\; \begin{array}{l} \overline{U}^{(N)}(\omega) \;\ge\; \nu_0^{(N)} - \frac{z_{1-\delta_1}}{\sqrt{m}}\,\tau_0^{(N)} \;\wedge \\ \overline{V}^{(N)}(\omega) \;\le\; \lambda_0^{(N)} + \frac{z_{1-\delta_2}}{\sqrt{m}}\,\vartheta_0^{(N)} \end{array}\right\}.$$

Thus, $A_N \in \Sigma$ holds for $N \in \mathbb{N}$ and

$$P(A_N) \;\ge\; \Phi\left(\frac{\lambda_0^{(N)} - \lambda^{(N)}}{\vartheta_0^{(N)}}\sqrt{m} + z_{1-\delta_2}\right) -$$

$$\Phi\left(\frac{\nu_0^{(N)} - \nu^{(N)}}{\tau_0^{(N)}}\sqrt{m} - z_{1-\delta_1}\right).$$

Define :

$$A \;=\; \lim_{N\to\infty}\sup A_N \;=\; \bigcap_{N=1}^{\infty}\bigcup_{K=N}^{\infty} A_K \;\in\; \Sigma$$

(see e.g. Henze [5]). We have :

$A \;=\; \{\omega \in \Omega \,|\, \omega \in A_N$ for an infinite number of N's$\}$.

Because of the definition $A'_N := \bigcup_{K=N}^{\infty} A_K$, it holds $A = \bigcap_{N=1}^{\infty} A'_N$, hence, we get for all $N \in \mathbb{N}$:

$$
\begin{aligned}
A'_N \supseteq A'_{N+1} &\implies A'_N \downarrow A \\
&\implies P\,(\lim_{N\to\infty} A'_N) = \lim_{N\to\infty} P\,(A'_N)\,.
\end{aligned}
$$

For all $N \in \mathbb{N}$, it holds :

$$
\begin{aligned}
P\,(\bigcup_{K=N}^{\infty} A_K) \geq P\,(A_N) \geq \Phi\left(\frac{\lambda_0^{(N)} - \lambda^{(N)}}{\vartheta_0^{(N)}}\sqrt{m} + z_{1-\delta_2}\right) - \\
\Phi\left(\frac{\nu_0^{(N)} - \nu^{(N)}}{\tau_0^{(N)}}\sqrt{m} - z_{1-\delta_1}\right)\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P\,(A) &= P\left(\bigcap_{N=1}^{\infty}\bigcup_{K=N}^{\infty} A_K\right) \\
&= \lim_{N\to\infty} P\,(A'_N) \\
&\geq \lim_{N\to\infty} P\,(A_N) \\
&\geq \Phi\left(\frac{\sup(\mu_0)_\alpha - \sup(\mu_0)_\alpha}{\sqrt{\mathrm{Var}\;\overline{Y}_\alpha}}\sqrt{m} + z_{1-\delta_2}\right) - \\
&\quad\; \Phi\left(\frac{\inf(\mu_0)_\alpha - \inf(\mu)_\alpha}{\sqrt{\mathrm{Var}\;\underline{Y}_\alpha}}\sqrt{m} - z_{1-\delta_1}\right) \quad \square
\end{aligned}
$$

Remark 6.2 :

For the case $\mu = \mu_0$, Lemma 6.1 implies : $L^\alpha(\mu_0) \geq 1 - (\delta_1 + \delta_2)$.

According to our stopping rule we have to carry out such a test simultaneously at N different levels α_i, $i = 1,\ldots,N$. Therefore, we define for every i, $i = 1,\ldots,N$ the statistics

$$\Psi_i[\overline{X}] := \begin{cases} 1 & \text{, if} \quad \mathrm{cl}(\overline{X})_{\alpha_i} \nsubseteq [\mathrm{UEG}_{\alpha_i}[\mu_0, \sigma_0], \mathrm{OEG}_{\alpha_i}[\mu_0, \sigma_0]], \\ 0 & \text{, otherwise,} \end{cases}$$

which leads to the test function

$$\Psi[\overline{X}] := \begin{cases} 1 & \text{, if} \quad \sum_{i=1}^{N} \Psi_i[\overline{X}] \geq K, \\ 0 & \text{, otherwise.} \end{cases}$$

From Lemma 6.1, it follows at once

$P(\Psi_i[\overline{X}] = 1) \leq 1 - L^{\alpha_i}(\mu)$ for all $i \in \{1, \dots, N\}$

and

$E(\Psi_i[\overline{X}]) \leq 1 - L^{\alpha_i}(\mu)$ for all $i \in \{1, \dots, N\}$.

Hence, for the operating charateristic function we get the following result.

Theorem 6.3 :

Under the assumptions of Lemma 6.1, it holds for every $N \in \mathbb{N}$, $K \in \{1, \dots, N\}$ and $\{\alpha_1, \dots, \alpha_N\} \subseteq [0, 1)$:

$$\begin{aligned} L(\mu) &:= P_\mu(\text{The process will not be stopped}) \\ &= P(H_0 : \mu = \mu_0 \text{ will be accepted if } \mu \text{ true parameter}) \\ &\geq 1 - \tfrac{1}{K} \sum_{i=1}^{N} (1 - L^{\alpha_i}(\mu)) \end{aligned}$$

Proof :

We have to show $P(\Psi[\overline{X}] = 1) \leq \frac{1}{K} \sum_{i=1}^{N} (1 - L^{\alpha_i}(\mu))$.

In order to proof this assertion, we use the following well-known inequality (see e.g. D.Morgenstern [16]) :

For every nonnegative random variable U and every $K \in \mathbb{N}$, it holds $P(U \geq K) \leq \frac{1}{K} E\,U$. Hence, we get

$$
\begin{aligned}
P(\Psi[\overline{X}] = 1) &= P(\sum_{i=1}^{n} \Psi_i[\overline{X}] \geq K) \\
&\leq \frac{1}{K} E\left(\sum_{i=1}^{N} \Psi[\overline{X}]\right) \\
&= \frac{1}{K} \sum_{i=1}^{N} E\left(\Psi[\overline{X}]\right) \\
&\leq \frac{1}{K} \sum_{i=1}^{N} (1 - L^{\alpha_i}(\mu))
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
L(\mu) &= P_\mu(\Psi[\overline{X}] = 0) \\
&= 1 - P_\mu(\Psi[\overline{X}] = 1) \\
&\geq 1 - \frac{1}{K} \sum_{i=1}^{N} (1 - L^{\alpha_i}(\mu)) \qquad \square
\end{aligned}
$$

Remark 6.4 :

In the case of $\mu = \mu_0$, it follows

$L(\mu_0) \geq 1 - \frac{1}{K} \sum_{i=1}^{N} (\delta_1 + \delta_2) = 1 - \frac{N}{K}(\delta_1 + \delta_2)$.

If we want to meet the following demands :

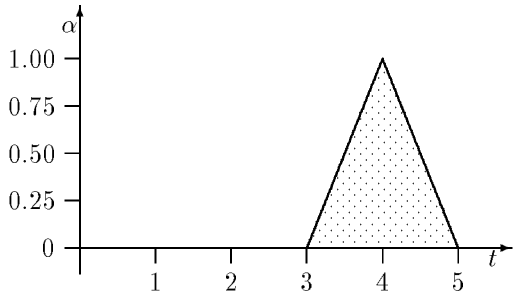
1. $\delta_1 \overset{!}{=} \delta_2$ (symmetry of the quantiles),
2. $L(\mu_0) \overset{!}{\geq} 1 - \delta$,

we get from $1 - \frac{N}{K}(\delta_1 + \delta_2) \overset{!}{=} 1 - \delta$ for δ_1 and δ_2 :

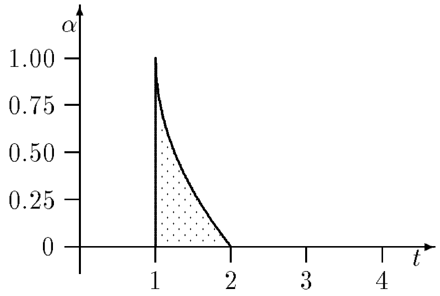
$\delta_1 = \delta_2 = \frac{\delta K}{2N}$, $\delta \in (0,1)$ $\qquad \square$

Example 6.5 :

Let the process variable $X : \Omega \to U_c(\mathbb{R})$ be a f.r.v. with normally distributed possible originals. We assume $E X = \mu_0$ und $\mathrm{Var}\, X = \sigma_0^2$ with



$$\mu_0(t) = \begin{cases} t-3 & , \text{if } \; 3 \le t \le 4 \\ 5-t & , \text{if } \; 4 \le t \le 5 \\ 0 & , \text{otherwise} \end{cases}$$



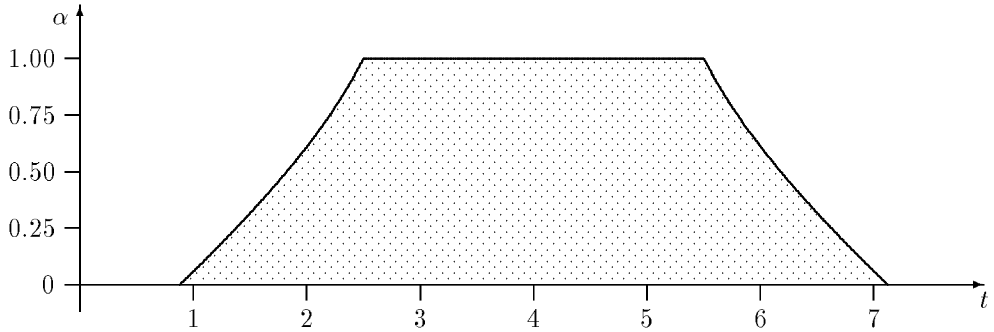
$$\sigma_0^2(t) = \begin{cases} (t-2)^2 & , \text{if } \; 1 \le t \le 2 \\ 0 & , \text{otherwise} \end{cases}$$

The control limits are given by

$$\begin{aligned} \mathrm{LCL}_\alpha &= \inf(\mu_0)_\alpha - \frac{z_{1-\frac{\delta k}{2N}}}{\sqrt{m}} \cdot \sqrt{\sup(\sigma_0^2)_\alpha}\,, \\ \mathrm{UCL}_\alpha &= \sup(\mu_0)_\alpha + \frac{z_{1-\frac{\delta k}{2N}}}{\sqrt{m}} \cdot \sqrt{\sup(\sigma_0^2)_\alpha}\,. \end{aligned}$$

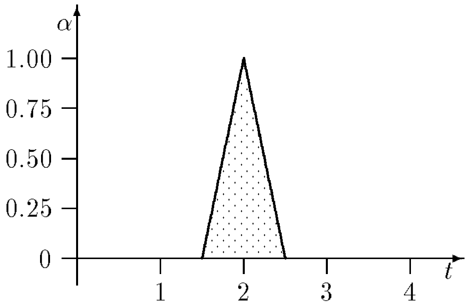
Choosing e.g. $m = 4$ and $z_{1-\frac{\delta k}{2N}} = 3.0$, we get $L(\mu_0) \ge 0.9973$.

The graphical representation of the fuzzy - set of the acceptance region is given below.In the in-control-state, the sample mean will be inside of this fuzzy - set with a probability of at least 0.9973.

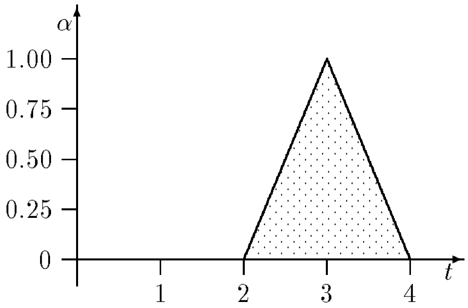


Let us consider the decision rule (control chart) with $N = 6$ and $K = 3$, where the six α-levels may be chosen as
$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} = \{0.1, 0.3, 0.5, 0.7, 0.8, 0.9\} \subseteq [0, 1)$.

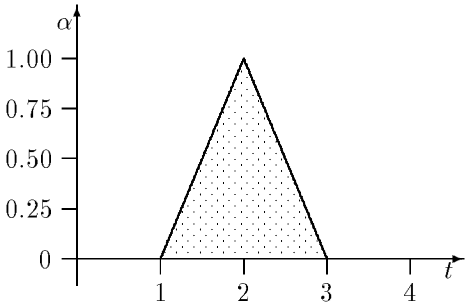
As an example, let us assume that we have observed the following fuzzy - random - sample of size 4 :



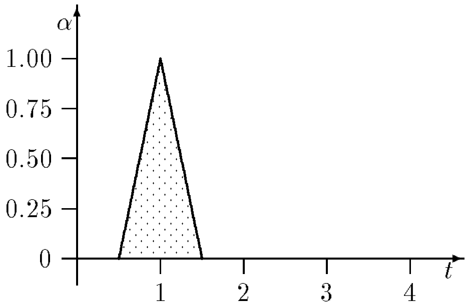
$$X_1(t) = \begin{cases} 2(t-\frac{3}{2}) & \text{, if } \frac{3}{2} \leq t \leq 2 \\ 2(\frac{5}{2}-t) & \text{, if } 2 \leq t \leq \frac{5}{2} \\ 0 & \text{, otherwise} \end{cases}$$



$$X_2(t) = \begin{cases} t-2 & \text{, if } 2 \leq t \leq 3 \\ 4-t & \text{, if } 3 \leq t \leq 4 \\ 0 & \text{, otherwise} \end{cases}$$

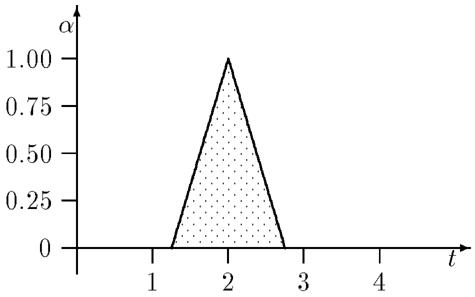


$$X_3(t) = \begin{cases} t-1 & \text{, if } 1 \leq t \leq 2 \\ 3-t & \text{, if } 2 \leq t \leq 3 \\ 0 & \text{, otherwise} \end{cases}$$



$$X_4(t) = \begin{cases} 2(t-\frac{1}{2}) & \text{, if } \frac{1}{2} \leq t \leq 1 \\ 2(\frac{3}{2}-t) & \text{, if } 1 \leq t \leq \frac{3}{2} \\ 0 & \text{, otherwise} \end{cases}$$

If we model our observations by this special class of fuzzy - sets (triangles, not necessarily symmetric), the sample mean can easily be computed. The graphical representation of the sample mean is naturally again a triangle. The center, as well as the left and right slope are the arithmetrical means of the centers, the left and the right slopes of the "sample triangles", respectively. In our example, we get :



$$\overline{X}(t) = \begin{cases} \frac{4}{3}(t - \frac{5}{4}) & \text{, if } \frac{5}{4} \le t \le 2 \\ \frac{4}{3}(\frac{11}{4} - t) & \text{, if } 2 \le t \le \frac{11}{4} \\ 0 & \text{, otherwise} \end{cases}$$

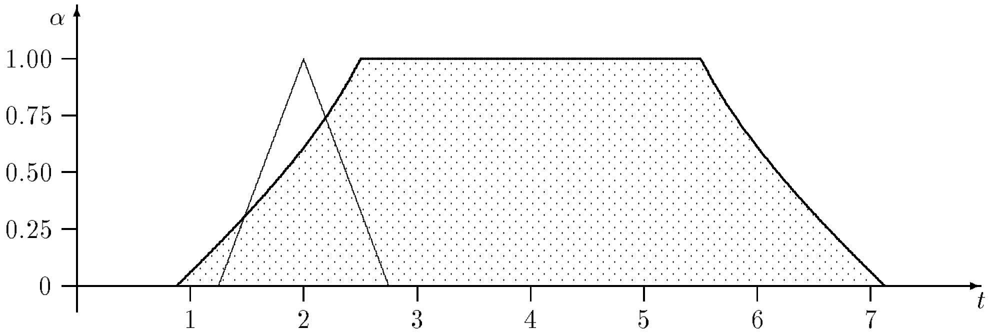
For the chosen α-levels, we give the corresponding α-cuts of the acceptance region and of the sample mean as well as the Ψ_i-values :

i	α_i	LCL_{α_i}	UCL_{α_i}	$\inf \overline{X}_{\alpha_i}$	$\sup \overline{X}_{\alpha_i}$	$\Psi_i(\overline{X})$
1	0.1	1.082	6.918	1.325	2.675	0
2	0.3	1.469	6.531	1.475	2.525	0
3	0.5	1.823	6.177	1.625	2.375	1
4	0.7	2.134	5.866	1.775	2.225	1
5	0.8	2.270	5.730	1.850	2.150	1
6	0.9	2.393	5.607	1.925	2.075	1

Because of $\sum_{i=1}^{6} \Psi_i(\overline{X}) = 4 \ge 3 = K$, we get for the test function $\Psi[\overline{X}]$:
$\Psi(\overline{X}) = 1$

Therefore, in this example the process will be stopped, our decision rule signals an out-of-control-state.

The graphical representation of the acceptance region and the observed sample mean may illustrate the situation at hand, and the need for α decision rule as suggested in this paper.



In a forthcoming paper we will study the statistical behaviour (e.g. average run length) of such fuzzy - Shewhart - charts.

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