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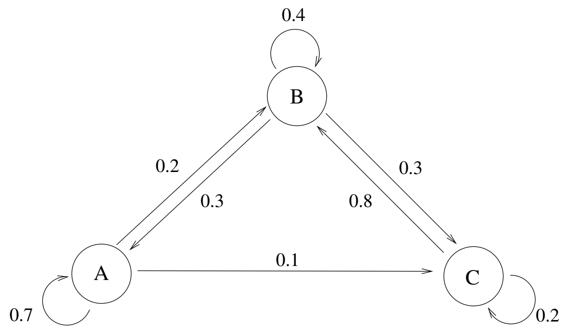
SS 2006
27.04.2006

Markov chains - Assignment 1

Exercise 1

(a) Set up a transition matrix corresponding to the state diagram:



(b) Draw a picture corresponding to this transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0.25 & 0.15 & 0.2 & 0.4 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0.3 & 0.4 & 0.1 & 0.2 \end{pmatrix}$$

(c) Look closely at state C in your picture of part b). What do you notice is strange about the way information flows near C? What effect do you think this might have on the long-range behaviour on this system?

Exercise 2

Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space $E = \{1, 2, 3\}$ and transition matrix $P = (p_{ij})$, where $p_{12} = p_{23} = p_{31} = 1$. The initial distribution is $\alpha = (1/3, 1/3, 1/3)$.

a) Show, that the following sequence does not define a Markov chain:

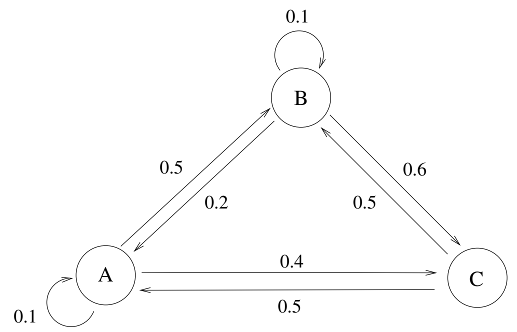
$$Y_n = \begin{cases} 0 & \text{if } X_n = 1, \\ 1 & \text{else.} \end{cases}$$

b) Set $Y_n = X_{2n}, n \geq 0$. Verify, that the sequence $\{Y_n\}$ is a Markov chain and determine the corresponding transition matrix.

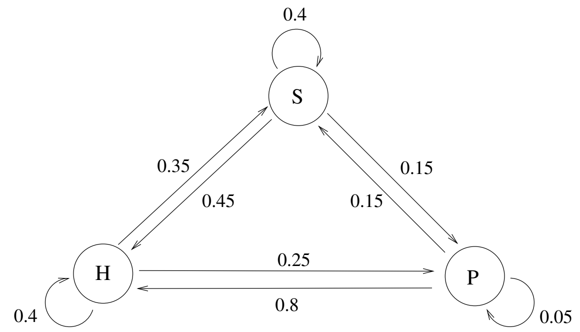
Exercise 3

Which of these situations can be modeled by a homogeneous Markov chain? If they cannot be modeled by a Markov chain, explain why not.

(a) The picture represents the probability that a delivery truck that is currently in region i (equal to A,B, or C) will be in region j for the next time period.



(b) The picture represents the probability that a person eating a meal i (equal to H = hamburger, S = salad or P = pizza) for lunch today will eat meal j (H,S, or P) for lunch tomorrow.



Exercise 4

Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_l)$ be the initial distribution and $\mathbf{P} = (p_{ij}), i, j \in E = \{1, ..., l\}$ the transition matrix. Furthermore, let $Z_0, Z_1, ...$ be a sequence of independent and identically distributed random variables uniformly distributed on $[0, 1]$. Define the E-valued random variable X_0 by

$$X_0 = k \;\Leftrightarrow\; Z_0 \in \left(\sum_{i=1}^{k-1} \alpha_i, \sum_{i=1}^{k} \alpha_i \right]$$

and the sequence $X_1, X_2, ...$ recursively by $X_n = \phi(X_{n-1}, Z_n)$, where $\phi : E \times [0, 1] \to E$ is definied as

$$\phi(i, z) = \sum_{k=1}^{l} k\mathbf{1} \left(\sum_{j=1}^{k-1} p_{ij} < z \leq \sum_{j=1}^{k} p_{ij} \right).$$

Verify that $\{X_n\}$ is a homogeneous Markov chain with initial distribution $\boldsymbol{\alpha}$ and transition martix $\mathbf{P}$.