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Markov chains - Assignment 2

Exercise 1

Consider a *cyclic random walk* $\{X_n, n \geq 0\}$ with state space $E = \{0, 1, ..., 9\}$ and (deterministic) initial state $x_0 = 5$. For $X_{n-1} = i$, let

$$X_n = \begin{cases} (i+1) \mod(10) & \text{with probability } 0.5, \text{ and} \\ (i-1) \mod(10) & \text{else.} \end{cases}$$

Calculate the following (conditional) probabilities:

(a) $P(X_3 \in \{2, 3, 4\})$,

(b) $P(X_3 = 6 \,|\, X_5 = 6)$,

(c) $P(X_5 = 6 \,|\, X_7 = 7, X_8 = 6)$.

Exercise 2

Suppose we live at a place where days are either summy, cloudy, or rainy. The weather is modelled a Markov chain with the following transition probabilities:

		tomorrow will be...		
		sunny	cloudy	rainy
	sunny	0.8	0.2	0.0
today it's...	cloudy	0.4	0.4	0.2
	rainy	0.2	0.6	0.2

(a) Suppose that the 10^{th} of May is a sunny day. Compute the probability distribution of the weather conditions on September 2^{nd}. Hint: Use the spectral representation of transition matrix P, i.e, $\mathbf{P}^n = \mathbf{\Phi}(\text{diag}(\theta))^n \mathbf{\Phi}^{-1}$, where $\mathbf{\Phi}$ is the matrix of Eigenvectors of $\mathbf{P}$, and $\text{diag}(\theta)$ is the diagonal matrix of the corresponding eigenvalues.

(b) Write a simulator that can randomly generate sequences of "weathers" from the given model. Hint: Use Exercise 4 of Assigment 1.

Exercise 3

Compute the n–step transition matrix $\mathbf{P}^{(n)}$ if $\mathbf{P} = \begin{pmatrix} 1-p & p \\ p' & 1-p' \end{pmatrix}, p, p' \in (0,1]$.

Exercise 4

A system used in automobile insurance is the so–called *bonus–malus system*. Here we consider the following:

- There exist four different premium classes $C_1, \ldots, C_4$ with insurance rates EUR 450, 350, 300 and 250, respectively.
- In the first year the policyholder is sorted in class C_1.
- The class of the policyholder in the following periods is determined by the class of the preceding period and the number of claims reported. If no claim is reported, his policy is upgraded to the next higher class. If one claim is reported in the current period, then he is downgraded by 1 level. Two reported claims cause a downgrade by 2 levels. If three or more claims are reported in one period, then the policyholder is sorted in class C_1.
- The number of claims reported in one period is modelled by a sequence of independent and identically $Poi(\lambda)$-random variables $\{Z_n\}$ for $\lambda = 2/3$.

Let X_n represent the class of the policyholder in the $(n+1)$-th period, $n \in \mathbb{N}_0$.

(a) Give a recursion equation for X_n of the form $X_n = \varphi(X_{n-1}, Z_n), n \geq 1$.

(b) Determine the corresponding transition matrix.

(c) Compute the expected premium value of a typical policyholder in the fourth period.