Prof. Dr. V. Schmidt / Dipl.-Math. oec. U. Pantle

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Markov chains - Assignment 3

Exercise 1

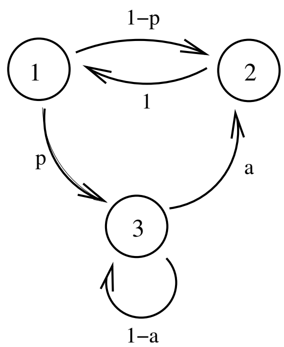
Let the transition matrix of a Markov chain be given as

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Is this Markov chain irreducible? Determine the periodicity of each state.

Exercise 2

Consider a Markov chain with the following transition graph:



(a) Determine all feasible values of a and p such that the Markov chain is irreducible or/and aperiodic.

(b) Compute the limit distribution $\pi^\top = (\pi_1, \pi_2\pi_3)$.

(c) For which values of a and p does it hold that $\pi_1 = \pi_2 = \pi_3$?

(d) Determine the mean recurrence time of state 2, that is:
$\mathbb{E}(\tau_2 \mid X_0 = 2)$ mit $\tau_2 = \min\{n > 0 : X_n = 2\}$.

Exercise 3

Let the Markov chain $\{X_n\}$ be ergodic with arbitrary initial distribution $\boldsymbol{\alpha}$ and limit distribution $\boldsymbol{\pi} = (\pi_1, ..., \pi_l)$. The function $h_j^{(n)} = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(X_k = j)$ indicates, how often the Markov chain $\{X_n\}$ visits state $j \in E$ on average in the first n transitions. Show that we have

$$\mathbb{E}_{\boldsymbol{\alpha}}(h_j^{(n)}) \to \pi_j \quad (n \to \infty)$$

and

$$P_{\boldsymbol{\alpha}}(|h_j^{(n)} - \pi_j| > \varepsilon) \to 0 \quad (n \to \infty)$$

(Hint: Make use of the fact that $p_{ij}^{(n)} = \pi_j + o_{ij}^{(n)}$ with $|o_{ij}^{(n)}| \leq br^n, 0 < r < 1, b \geq 0$.)