Prof. Dr. V. Schmidt / Dipl.-Math. oec. U. Pantle — SS 2006

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Markov chains - Assignment 4

Exercise 1

Consider a birth and death process with two reflecting barriers, i.e., $E = \{0, 1, \ldots, \ell\}$ and

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & & & & & & \\ q_1 & r_1 & p_1 & & & & & \\ & q_2 & r_2 & p_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & q_i & r_i & p_i & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & q_{\ell-1} & r_{\ell-1} & p_{\ell-1} \\ & & & & & & 1 & 0 \end{pmatrix}.$$

(a) Check that the vector $\boldsymbol{\alpha}$ with

$$\alpha_i = \alpha_0 \, \frac{p_1 p_2 \cdot \ldots \cdot p_{i-1}}{q_1 q_2 \cdot \ldots \cdot q_i}, \qquad i = 1, \ldots, \ell$$

and

$$\alpha_0 = \left(1 + \frac{1}{q_1} + \frac{p_1}{q_1 q_2} + \ldots + \frac{p_1 p_2 \cdot \ldots \cdot p_{\ell-1}}{q_1 q_2 \cdot \ldots \cdot q_\ell}\right)^{-1}.$$

satisfies the system of linear equations $\boldsymbol{\alpha}^\top = \boldsymbol{\alpha}^\top \mathbf{P}$.

(b) Show explicitly that the given Markov chain is irreducible and aperiodic if $r_i > 0$ for some $i \in \{0, \ldots, \ell\}$. What happens if $r_i = 0$ for all $i \in \{0, \ldots, \ell\}$?

Exercise 2

Let $\mathbf{P}$ be a doubly stochastic transition matrix on $E = \{1, \ldots, \ell\}$ (that is to say: $\mathbf{P}^\top$ is also a stochastic matrix).

(a) Prove that the uniform distribution on E is always a stationary initial distribution.

(b) Find an example, where the uniform distribution is the unique stationary initial distribution, and an example, where also other stationary initial distributions exist.

Exercise 3

Let $\{X_n\}$ be an ergodic Markov chain with transition matrix $\mathbf{P}$ and limit distribution $\boldsymbol{\pi}^\top = (\pi_1, \ldots, \pi_\ell)$. Denote by $\tau_j^+ = \inf\{n \geq 1 : X_n = j\}$ the time of the first visit to j (called the *first-hitting time*), and define $\mu_{ij} = \mathbb{E}(\tau_j^+ \mid X_0 = i)$ (hint: since $\{X_n\}$ is ergodic, we have $\mu_{ij} < \infty$) for $i, j \in \{1, \ldots, \ell\}$. Furthermore, write $\mathbf{E} = (1)_{i,j=1,\ldots,\ell}$. Verify the following claims:

(a) Let $\mathbf{M} := (\mu_{ij})_{i,j \in E}$, then $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{P}(\mathbf{M} - \mathbf{M}_{\text{diag}}) + \mathbf{E}$, where $\mathbf{M}_{\text{diag}} = \text{Diag}(\mu_{11}, ..., \mu_{ll})$.

(b) For all states $i \in \{1, \ldots, \ell\}$, it holds that $\mu_{ii} = {\pi_i}^{-1}$.

(c) There exists exactly one matrix $\mathbf{M}$ that satisfies the equality of part (a).

(d) The matrix $\mathbf{M}$ of mean first-hitting times is given by

$$\mathbf{M} = (\mathbf{I} - \mathbf{Z} + \mathbf{E}\mathbf{Z}_{\text{diag}})\mathbf{D}$$

with $\mathbf{Z} = (\mathbf{I} - (\mathbf{P} - \mathbf{\Pi}))^{-1} = \mathbf{I} + \sum_{n=1}^{\infty} (\mathbf{P}^n - \mathbf{\Pi})$ and $\mathbf{D} = \text{Diag}(\frac{1}{\pi_1}, ..., \frac{1}{\pi_\ell})$.

Hint: Show that $(\mathbf{I} - \mathbf{P})\mathbf{Z} = \mathbf{I} - \mathbf{\Pi}$.

Exercise 4

A (six–sided) dice is repeatedly thrown. The outcome of each roll is repesented by the random variables $X_1, X_2, ...$

Let $S_n = X_1 + ... + X_n$ and

$$T_1 = \min\{n \geq 1 : S_n \text{ is divisible by eight}\}$$

$$T_2 = \min\{n \geq 1 : S_n - 1 \text{ is divisible by eight}\}.$$

Determine $\mathbb{E}T_1$ and $\mathbb{E}T_2$ by means of Exercise 3 (b) and (d).