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SS 2006
30.06.2006

Markov chains - Assignment 6

Exercise 1

Normally distributed random variables can be simulated using the so–called Box-Muller algorithms: a pair (U_1, U_2) of independent $(0,1]$-uniformly distributed random variables is transformed into (X_1, X_2) by means of

$$X_1 = \sqrt{-2\log U_1}\cos(2\pi U_2) \qquad X_2 = \sqrt{-2\log U_1}\sin(2\pi U_2).$$

Prove that X_1 and X_2 are independent and identically $N(0,1)$-distributed.
Use this result, to obtain a sample of $N(0.5, 9)$-distributed random variables using a LCG with parameters $m = 2147483647, a = 48271, c = 0$ and $z_0 = 1$. Determine the empirical mean and variance of the first $n = 100, 1000$ resp. 10000 pseudo random numbers.

Exercise 2

For each of the following densities, write down an algorithm based on the acceptance–rejection method to generate pseudo random numbers according to each of the given distributions – call it G (with density q). At this, assume that the only random number generator currently at hand produces $U(0,1)$-pseudo random numbers. With regard to parts (b) and (c): The auxiliary distribution F (with density p), from which candidates for the realizations from G are drawn, should be simple to generate, but not too far-off from the desired distribution.

(a) $q_j = \dfrac{a}{j}, \quad j = 1, 2, ..., 100$, with $a = \left(\sum_{j=1}^{100} q_j\right)^{-1}$,

(b) $q(y) = \dfrac{1}{10}y^2 + \dfrac{7}{15}$ for $y \in (-1, 1)$,

(c) $q(y) = \dfrac{3}{4}(1 + y^2)$ for $y \in (0, 1)$.

Exercise 3

A random variable with density $q(y) = \sqrt{2\pi^{-1}}e^{-y^2/2}$, $y \geq 0$ is to be simulated by rejection sampling. The candidate values are realizations from a $Exp(\lambda)$-distributed random variable, i.e., $p(x) = \lambda e^{-\lambda x}, x \geq 0$.

(a) Determine the smallest value c (subject to $\lambda > 0$) such that $c \cdot p(y) \geq q(y)$.

(b) For which value of λ, is the theoretically percentage of rejected samples minimal?

(c) For $\lambda = 1$, write a simulator to generate pseudo random numbers according to the stated setup. Determine the theoretical percentage of rejected values and compare it to the empirical result for $n = 1000$ iterations.