Queueing Systems manuscript No.
(will be inserted by the editor)

On the distribution of typical shortest–path lengths in connected random geometric graphs

D. Neuhäuser, C. Hirsch, C. Gloaguen, V. Schmidt

the date of receipt and acceptance should be inserted later

Abstract Stationary point processes in $\mathbb{R}^2$ with two different types of points, say H and L, are considered where the points are located on the edge set G of a random geometric graph, which is assumed to be stationary and connected. Examples include the classical Poisson–Voronoi tessellation with bounded and convex cells, aggregate Voronoi tessellations induced by two (or more) independent Poisson processes whose cells can be non–convex, and so–called β-skeletons being subgraphs of Poisson–Delaunay triangulations. The length of the shortest path along G from a point of type H to its closest neighbor of type L is investigated. Two different meanings of 'closeness' are considered: either with respect to the Euclidean distance (e-closeness), or in a graph–theoretic sense, i.e., along the edges of G (g-closeness). For both scenarios, comparability and monotonicity properties of the corresponding typical shortest–path lengths C^{e*} and C^{g*} are analyzed. Furthermore, extending the results which have recently been derived for C^{e*}, we show that the distribution of C^{g*} converges to simple parametric limit distributions if the edge set G becomes unboundedly sparse or dense, i.e., a scaling factor κ converges to zero and infinity, respectively.

Keywords Point Process, Aggregate Tessellation, β-Skeleton, Shortest Path, Palm Mark Distribution, Stochastic Monotonicity, Scaling Limit

D. Neuhäuser, C. Hirsch, V. Schmidt
Institute of Stochastics, Ulm University, 89069 Ulm, Germany
Tel.: +49-731-5023532
E-mail: volker.schmidt@uni-ulm.de

C. Gloaguen
Orange Labs, 38-40, rue du Général Leclerc, 92794 Issy-les-Moulineaux, France
Tel.: +33-(0)1 45 29 64 41
E-mail: catherine.gloaguen@orange-ftgroup.com

1 Introduction

In this paper we extend results on distributional properties of typical shortest–path lengths in spatial stochastic network models which have recently been derived in [8, 17, 18]. More precisely, we consider stochastic models for networks with two hierarchy levels, i.e., there are network components of two different kinds: high–level components (HLC) and low–level components (LLC). The locations of both HLC and LLC are represented by points on the edge set G of a random geometric graph ([13]) in $\mathbb{R}^2$ which is assumed to be stationary and connected. It is clear that this is fulfilled for the edge set of stationary tessellations with convex cells (see e.g. [14, 15]). But, for instance, also for so–called aggregate tessellations ([2, 16]) and β-skeletons ([1, 10]) induced by homogeneous Poisson processes.

Each LLC is assumed to be connected to its closest HLC, where two different meanings of 'closeness' are considered: either with respect to the Euclidean distance (e-closeness), or in a graph–theoretic sense, i.e., along the edges of the graph (g-closeness). In applications, e.g. to telecommunication networks, the edges of the random geometric graph can represent the underlying infrastructure, for instance, an inner–city street system. In this case, one is especially interested in the distribution of shortest–path lengths along the edge set between the LLC and their closest HLC, which is an important performance characteristic in cost and risk analysis as well as in strategic planning of wired telecommunication.

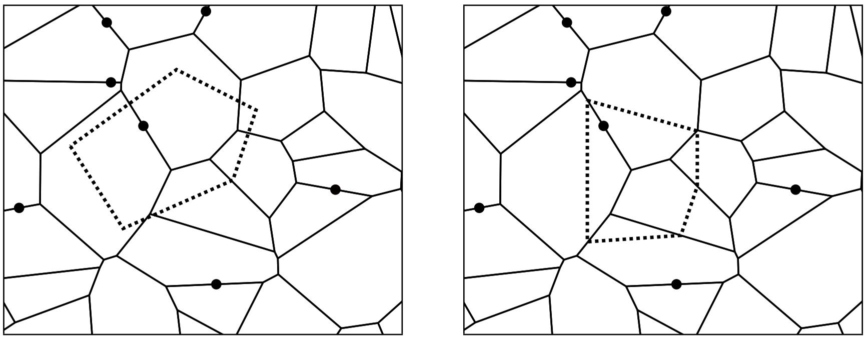

Fig. 1 Edge set of a Voronoi tessellation with locations of HLC and serving zones (dashed) for e-closeness (left) and g-closeness (right)

In [8, 17, 18], we associated with each HLC a certain subset of $\mathbb{R}^2$ which is called its serving zone. Each LLC was linked to the HLC in whose serving zone the LLC is located. In particular, we assumed that the serving zones were constructed as the cells of the Voronoi tessellation with respect to the locations of HLC. This is equivalent to link each LLC to its e-closest HLC. In the present paper, we additionally study a modified connection rule where we replace the Voronoi cells, considered so far for e-closeness, by Voronoi cells constructed with respect to the graph metric (g-closeness), see Figure 1. That is, each LLC is connected to its g-closest HLC, i.e., the HLC to which the distance measured along the edges of the graph is smallest.

We assume that the locations of HLC and LLC are given by two stationary Coxian point processes X_H and X_L in $\mathbb{R}^2$ (see e.g. [5,9,15]) whose random intensity measures are concentrated on G. In particular, we assume that (i) X_H and X_L are conditionally independent given G and (ii) their random intensity measures are proportional to the one–dimensional Hausdorff measure on G, with some (linear) intensities $\lambda_\ell, \lambda'_\ell > 0$ for X_H and X_L, respectively.

In this case, one is especially interested in the distributions of the typical shortest–path lengths C^{e*} and C^{g*} along the edge set between the points of X_L and their e-closest resp. g-closest neighbors in X_H. Note that even for simple examples of stationary and fully connected edge sets G in $\mathbb{R}^2$, the distributions of C^{e*} and C^{g*} are not known analytically. However, asymptotic results can be derived if the edge set G becomes unboundedly sparse or dense. In particular, under some additional conditions, it can be shown that the distributions of C^{e*} and C^{g*} converge to exponential and Weibull distributions, respectively, which do not depend on the selected closeness scenario. Furthermore, it can be shown that the distribution of C^{g*} does not depend on λ'_ℓ and, for g-closeness, that it decreases stochastically in λ_ℓ, i.e., the values of the distribution function of C^{g*} increase pointwise if λ_ℓ increases. Although increasing the linear intensity λ_ℓ of X_H certainly decreases the size of the typical serving zone for both e-closeness and g-closeness, it seems to be an open problem whether in the case of e-closeness the distribution of C^{e*} decreases stochastically in λ_ℓ. However, our numerical experiments clearly indicate that this should be true, see Section 6. On the other hand, it can be shown that C^{g*} under g-closeness is stochastically smaller than C^{e*} under e-closeness.

The paper is organized as follows. In Section 2 we describe the hierarchical network model investigated in the present paper. Examples of random geometric graphs, which are stationary and connected, are discussed in Section 3. Then, Section 4 deals with comparability and monotonicity properties of the distributions of typical shortest–path lengths C^{e*} and C^{g*}. Afterwards, in Section 5, we investigate the asymptotic behavior of these distributions for unboundedly sparse and dense networks, respectively. In Section 6, we present some numerical results. Finally, Section 7 concludes the paper and gives an outlook to future research.

2 Stochastic modelling of hierarchical networks

To begin with we give a short description of the hierarchical network model investigated in the present paper. In particular, we briefly introduce some fundamental classes of models from stochastic geometry which we are using in order to construct the network model. The reader, who is interested in further details, is referred to well–known monographs, see e.g. [5,9,15] for random (marked) point processes and, in particular, Coxian point processes, [13] for random geometric graphs, [14,15] for random tessellations, and [11,14] for general random closed sets.

2.1 Marked point processes

First we recall some basic notions regarding marked point processes in $\mathbb{R}^2$. Let $\mathcal{B}^2$ denote the family of Borel sets of $\mathbb{R}^2$ and let $\mathbb{M}$ be a Polish space with its Borel σ-algebra $\mathcal{B}_{\mathbb{M}}$. Furthermore, for any $n \geq 1$, let $X_n : \Omega \to \mathbb{R}^2$ and $M_n : \Omega \to \mathbb{M}$ be random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\#\{n : X_n \in B\} < \infty$ with probability 1 for each bounded $B \in \mathcal{B}^2$. Then, $X = \{(X_n, M_n),\, n \geq 1\}$ is said to be a marked point process with mark space $\mathbb{M}$.

Note that we can regard X as a random element of $(N_{\mathbb{M}}, \mathcal{N}_{\mathbb{M}})$, where $N_{\mathbb{M}}$ is the family of all counting measures on $\mathcal{B}^2 \otimes \mathcal{B}_{\mathbb{M}}$ which are simple and locally finite in the first component and $\mathcal{N}_{\mathbb{M}}$ is the usual σ-algebra on $N_{\mathbb{M}}$. We thus can regard a (marked) point process X as a random counting measure, i.e. $X = \{X(B \times E),\, B \in \mathcal{B}^2, E \in \mathcal{B}_{\mathbb{M}}\}$, where

$$X(B \times E) = \#\{n : X_n \in B, M_n \in E\}\,.$$

For $x \in \mathbb{R}^2$ we define the shift $\mathbf{t}_x : N_{\mathbb{M}} \to N_{\mathbb{M}}$ by $\mathbf{t}_x X = \mathbf{t}_x\{(X_n, M_n)\} = \{(X_n - x, M_n)\}$. Assume now that $X = \{(X_n, M_n)\}$ is stationary with intensity $\lambda \in (0, \infty)$, i.e., $X \stackrel{\mathrm{d}}{=} \mathbf{t}_x X$ holds for each $x \in \mathbb{R}^2$, where $\stackrel{\mathrm{d}}{=}$ means equality of distributions; $\lambda = \mathbb{E}\#\{n : X_n \in [0,1)^2\}$. Then the Palm mark distribution $\mathbb{P}_X^o : \mathcal{B}_{\mathbb{M}} \to [0,1]$ of X is given by

$$\mathbb{P}_X^o(E) = \frac{1}{\lambda}\, \mathbb{E}\#\{n : X_n \in [0,1)^2, M_n \in E\}\,, \qquad E \in \mathcal{B}_{\mathbb{M}}\,. \tag{1}$$

A random variable M^* distributed according to $\mathbb{P}_X^o$ is called the typical mark of X.

In the following, two jointly stationary marked point processes $X^{(1)} = \{(X_n^{(1)}, M_n^{(1)})\}$ and $X^{(2)} = \{(X_n^{(2)}, M_n^{(2)})\}$ with intensities λ_1 and λ_2 and mark spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, will be considered as a random element $Y = (X^{(1)}, X^{(2)})$ of the product space $N_{\mathbb{M}_1,\mathbb{M}_2} = N_{\mathbb{M}_1} \times N_{\mathbb{M}_2}$. The Palm distribution $\mathbb{P}^*_{X^{(i)}}$ of Y with respect to the i-th component, $i = 1, 2$, is then defined on $\mathcal{N}_{\mathbb{M}_1} \otimes \mathcal{N}_{\mathbb{M}_2} \otimes \mathcal{B}_{\mathbb{M}_i}$ by

$$\mathbb{P}^*_{X^{(i)}}(A \times E) = \frac{1}{\lambda_i}\, \mathbb{E}\#\{n : X_n^{(i)} \in [0,1)^2, M_n^{(i)} \in E, \mathbf{t}_{X_n^{(i)}} Y \in A\}\,, \tag{2}$$

where $A \in \mathcal{N}_{\mathbb{M}_1} \otimes \mathcal{N}_{\mathbb{M}_2}$ and $E \in \mathcal{B}_{\mathbb{M}_i}$. Note that the Palm mark distribution $\mathbb{P}^o_{X^{(i)}}$ of $X^{(i)}$ can be obtained from $\mathbb{P}^*_{X^{(i)}}$ as a marginal distribution.

2.2 Random geometric graphs

The edge set of a random geometric graph can be described by an $\mathcal{F}$–valued random variable $G : \Omega \to \mathcal{F}$ such that $\mathbb{P}(G \in \mathcal{S}) = 1$, where $\mathcal{F}$ denotes the family of all closed subsets of $\mathbb{R}^2$ and $\mathcal{S} \subset \mathcal{F}$ is the family of all locally finite unions of bounded closed segments. The random edge set G is called stationary if $\mathbb{P}(G \neq \emptyset) = 1$ and $G \stackrel{\mathrm{d}}{=} G + x$ holds for each $x \in \mathbb{R}^2$. If G is stationary, then we define the intensity γ of G as the expected total edge length per unit area, i.e., $\gamma = \mathbb{E}\nu_1(G \cap [0,1]^2)$, where ν_1 denotes the 1-dimensional Hausdorff measure. Furthermore, G is said to be connected if

(i) for any pair $e, e' \in G$ of random segments with $e \neq e'$, the set $e \cap e'$ is either empty or consists of a common endpoint of e and e',
(ii) for any pair $e, e' \in G$ of random segments, there exists a (random) integer $n \geq 1$ and a sequence $e_1, \ldots, e_n \in G$ of random segments such that

$$e \cap e_1 \neq \emptyset, e_1 \cap e_2 \neq \emptyset, \ldots, e_{n-1} \cap e_n \neq \emptyset, e_n \cap e' \neq \emptyset\,.$$

If, in addition to conditions (i) and (ii), it holds that

(iii) $G' = \bigcup_{n=1}^{\infty} \partial\Xi_n$ for some random subset $G' \subset G$ of edges, where $\Xi_1, \Xi_2, \ldots : \Omega \to \mathcal{F}$ are bounded (but not necessarily convex) random polygons such that $\bigcup_{n=1}^{\infty} \Xi_n = \mathbb{R}^2$, $\operatorname{int}\Xi_i \neq \emptyset$ and $\operatorname{int}\Xi_i \cap \operatorname{int}\Xi_j = \emptyset$ for any $i, j \geq 1$ with $i \neq j$, and $\#\{n : \Xi_n \cap B \neq \emptyset\} < \infty$ for each bounded $B \in \mathcal{B}^2$, where $\operatorname{int}\Xi$ denotes the inner part of the set Ξ,

then G is said to be fully connected.

Let $G' \subset G$ be the maximum subset of G which satisfies the conditions mentioned in (iii). Then, G' can be seen as a random tessellation of the Euclidean plane with bounded but not necessarily convex cells, whereas the random set $G \setminus \operatorname{int} G'$ can be interpreted as a family of 'dead ends', see Figure 2.

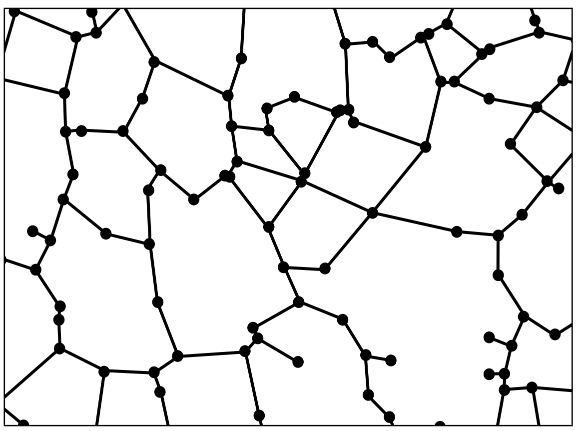

Fig. 2 Tessellation with bounded but not necessarily convex cells and dead ends

In the following we always assume that the random edge set G is stationary with $\mathbb{E}\nu_1(G \cap [0,1]^2) = 1$. Furthermore, for each $\gamma > 0$ we consider the scaled edge set G_γ with intensity γ defined by $G_\gamma = G/\gamma$, i.e., we scale the edge set G such that $\mathbb{E}\nu_1(G_\gamma \cap [0,1]^2) = \gamma$.

2.3 Serving zones and shortest path lengths

For any $\gamma > 0$, we consider stationary Cox point processes $X_H = \{X_{H,n}\}$ and $X_L = \{X_{L,n}\}$ whose random intensity measures are concentrated on the scaled edge set $G_\gamma = G/\gamma$, where we assume that G satisfies the connectivity conditions (i) and (ii) introduced in Section 2.2. The Cox processes $X_H$and X_L are used in order to model the locations of HLC and LLC. In particular, we assume that (i) X_H and X_L are conditionally independent given G_γ and (ii) their random intensity measures are proportional to the one–dimensional Hausdorff measure ν_1 on G_γ, i.e., $\mathbb{E}X_H(B) = \lambda_\ell\, \mathbb{E}\nu_1(B \cap G_\gamma)$ and $\mathbb{E}X_L(B) = \lambda'_\ell\, \mathbb{E}\nu_1(B \cap G_\gamma)$ for each Borel set $B \in \mathcal{B}^2$ and for some (linear) intensities $\lambda_\ell, \lambda'_\ell > 0$. Thus, the Cox processes X_H and X_L can be constructed by placing homogeneous Poisson processes on the edges of G_γ with linear intensity λ_ℓ and λ'_ℓ, respectively. Note that the planar intensities λ and λ' of X_H and X_L are given by $\lambda = \lambda_\ell\gamma$ and $\lambda' = \lambda'_\ell\gamma$.

Each LLC is assumed to be connected to its closest HLC, where two different notions of closeness are considered: either with respect to the Euclidean distance (e-closeness), or in a graph–theoretic sense, i.e., along the edges of the graph (g-closeness), see Figure 1.

In the first case, we consider the Voronoi tessellation $T_H = \{\Xi_{H,n}\}$ induced by the points $X_{H,n}$ of the Cox process $X_H = \{X_{H,n}\}$, i.e.

$$\Xi_{H,n} = \{x \in \mathbb{R}^2 : |x - X_{H,n}| \leq |x - X_{H,m}| \text{ for all } m \neq n\},$$

where $|\cdot|$ denotes the Euclidean norm. The Voronoi cell $\Xi_{H,n}$ is considered to be the serving zone of the HLC located at $X_{H,n}$. Furthermore let us denote by $S^e_{H,n} = (G_\gamma \cap \Xi_{H,n}) - X_{H,n}$ the segment system of the serving zone $\Xi_{H,n}$ corresponding to $X_{H,n}$, centered at the origin o.

We can then construct the stationary marked point process $X_{L,C^e} = \{(X_{L,n}, C^e_n)\}$, where the mark C^e_n is the length of the shortest path from $X_{L,n}$ to $X_{H,j}$ along the edge set G_γ provided that $X_{L,n} \in \Xi_{H,j}$. Thus, each LLC is connected to that HLC which is closest to it in the Euclidean sense.

In the second case, the segment system centered at o of the serving zone of $X_{H,n}$, denoted by $S^g_{H,n}$, corresponds to the Voronoi cell $S^g_{H,n}$ in the graph metric, i.e.,

$$S^g_{H,n} = \{x \in G_\gamma : c(x, X_{H,n}) \leq c(x, X_{H,m}) \text{ for all } m \geq 1\} - X_{H,n}\,,$$

where $c(x, X_{H,n})$ denotes the length of the shortest path from $x \in G_\gamma$ to $X_{H,n}$ along the edge set G_γ. Similar as above, we regard the stationary marked point process $X_{L,C^g} = \{(X_{L,n}, C^g_n)\}$, where the mark $C^g_n = \min_{m\geq 1}(c(X_{L,n}, X_{H,m}))$

is the minimal length of shortest paths from $X_{L,n}$ to the points of X_H along the edges of G_γ. Thus, each LLC is connected to that HLC which is closest to it in a graph–theoretic sense.

3 Examples

3.1 Stationary tessellations with convex cells

As an example of a random geometric graph, whose edge set G is fully connected, we consider the edge set of a random tessellation $T = \{\Xi_n, n \geq 1\}$ of $\mathbb{R}^2$ with convex cells, i.e. $G = G' = \bigcup_{n=1}^{\infty} \partial \Xi_n$, where $\Xi_1, \Xi_2, \ldots : \Omega \to \mathcal{F}$ are bounded and convex random polygons such that $\bigcup_{n=1}^{\infty} \Xi_n = \mathbb{R}^2$, $\operatorname{int} \Xi_i \neq \emptyset$ and $\operatorname{int} \Xi_i \cap \operatorname{int} \Xi_j = \emptyset$ for any $i, j \geq 1$ with $i \neq j$, and $\#\{n : \Xi_n \cap B \neq \emptyset\} < \infty$ for each bounded $B \in \mathcal{B}^2$.

Special emphasis will be put on three classes of stationary tessellations, which are induced by homogeneous Poisson point processes: isotropic Poisson line tessellations (PLT), Poisson–Voronoi tessellations (PVT), and Poisson–Delaunay tessellations (PDT), see also [8,17,18]. Note that the edge set of a PVT is shown in Figure 1.

3.2 Aggregate tessellations

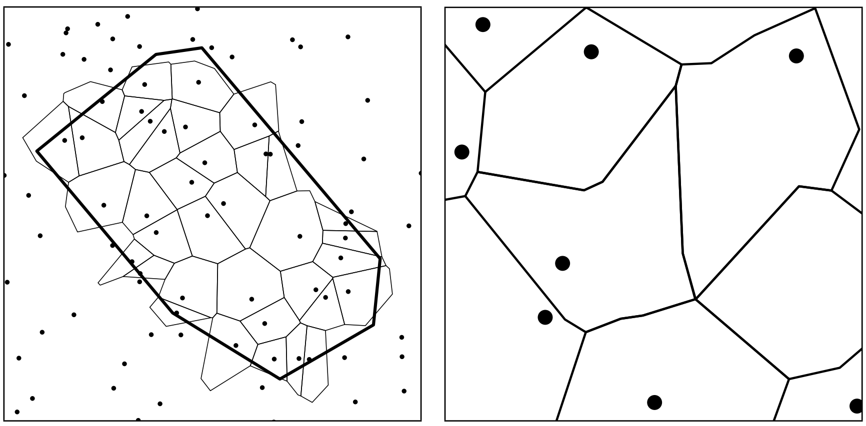

Fig. 3 Construction principle (left) and realization (right) of an PVAT with bounded non-convex cells

Stationary tessellations with bounded, but not necessarily convex cells can be constructed e.g. in the following way, see [2,16]. For example, let $X^{(1)} = \{X_n^{(1)}\}$ and $X^{(2)} = \{X_n^{(2)}\}$ be two independent homogeneous Poisson processes. Let $T^{(1)} = \{\Xi_n^{(1)}\}$ and $T^{(2)} = \{\Xi_n^{(2)}\}$ denote the PVT induced by $X^{(1)}$ and $X^{(2)}$, respectively. Then, the sequence of random closed sets $T = \{\Xi_n\}$ with

$$\Xi_n = \bigcup_{i: X_i^{(2)} \in \Xi_n^{(1)}} \Xi_i^{(2)} \tag{3}$$

is called an aggregate Poisson–Voronoi tessellation (PVAT) induced by $X^{(1)}$ and $X^{(2)}$. It is not difficult to see that the random edge set $G = \bigcup_{n=1}^{\infty} \partial \Xi_n$ induced by the cells Ξ_n given in (3) is stationary and fully connected, see Figure 3. Furthermore, an PVAT does not have dead ends, i.e., $G = G'$.

3.3 β-skeletons

Another interesting class of geometric graphs with connected edge set is formed by β-skeletons, first introduced in [10]. Let $\beta \in [1, 2]$ and let $I \subset \mathbb{R}^d$ be a locally finite set. The β-skeleton $G(\beta, I)$ is the edge set of a graph with vertex set I, which is defined as follows. For $x, y \in I$ let

$$m_{xy}^{(1)} = \frac{\beta}{2}\, x + (1 - \frac{\beta}{2})\, y\,, \qquad m_{xy}^{(2)} = (1 - \frac{\beta}{2})\, x + \frac{\beta}{2}\, y\,,$$

and

$$A_\beta(x, y) = B(m_{xy}^{(1)}, |m_{xy}^{(1)} - y|) \cap B(m_{xy}^{(2)}, |m_{xy}^{(2)} - x|)\,, \tag{4}$$

where $B(z, r) = \{z' \in \mathbb{R}^2 : |z - z'| < r\}$ denotes the open ball with center at $z \in \mathbb{R}^2$ and with radius $r \geq 0$, see Figure 4. Then we put

$$G(\beta, I) = \bigcup_{x, y \in I:\, I \cap A_\beta(x,y) = \emptyset} [x, y]\,. \tag{5}$$

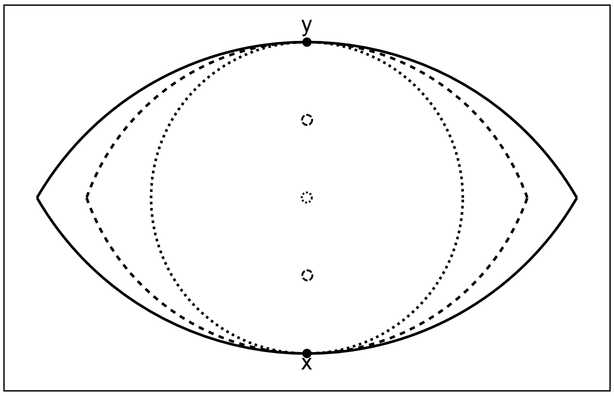

Fig. 4 Schematic representation of the intersection of balls $A_\beta(x, y)$ for $\beta = 1$ (dotted), $\beta = 1.5$ (dashed) and $\beta = 2$ (solid)

Note that the edge set $G(\beta, I)$ is monotonously decreasing in β, see Figure 5. Furthermore, it is not difficult to see that β-skeletons $G(\beta, I)$ on a finite vertex set I always fulfill the connectivity conditions (i) and (ii) introduced in Section 2.2. Property (i) immediately follows from the definition of β-skeletons given in (4) and (5). Moreover, for any finite vertex set I the following relationship holds: For any $\beta \in [1,2]$, we have

$$\mathrm{MST}(I) \subseteq \mathrm{RNG}(I) = G(2,I) \subseteq G(\beta,I)\,, \qquad (6)$$

where $\mathrm{MST}(I)$ and $\mathrm{RNG}(I)$ denote the minimum spanning tree and relative neighborhood graph of I, respectively (see e.g. [3]). Since $MST(I)$ fulfills condition (ii) by definition, (6) implies that this is true also for $G(\beta,I)$. Note however that for an arbitrary point process $X \subset \mathbb{R}^d$ proving a.s. connectivity or the finiteness of cells of $G(\beta,X)$ is a non-trivial problem.

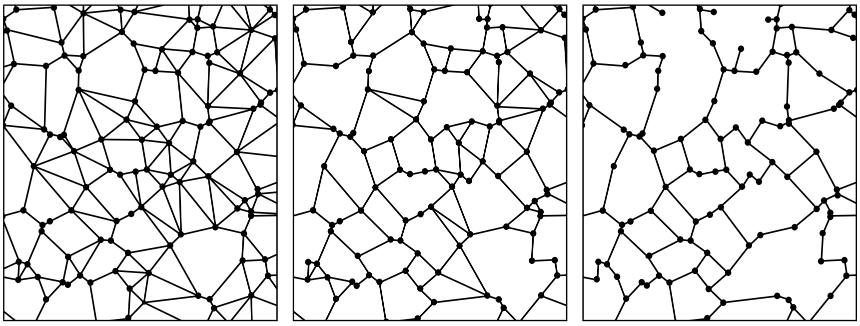

Fig. 5 Examples of β-skeletons for $\beta = 1$ (left), $\beta = 1.5$ (center), and $\beta = 2$ (right)

4 Comparability and monotonicity

Let $G_\gamma = G/\gamma$ be an arbitrary stationary random edge set in $\mathbb{R}^2$ with some intensity $\gamma > 0$ which is connected, i.e. G satisfies the connectivity conditions (i) and (ii) introduced in Section 2.2. We now define the typical shortest–path lengths $C^{e*} = C^{e*}(\gamma, \lambda_\ell, \lambda'_\ell)$ and $C^{g*} = C^{g*}(\gamma, \lambda_\ell, \lambda'_\ell)$ and show that (i) C^{g*} is stochastically smaller than C^{e*} for all $(\gamma, \lambda_\ell, \lambda'_\ell) \in [0,\infty)^3$, and (ii) C^{g*} is stochastically decreasing if λ_ℓ increases. We conjecture that the monotonicity property (ii) is also true for C^{e*}, as indicated by the numerical results displayed in Figure 7.

What we are mainly interested in are the distributions of the typical marks C^{e*} and C^{g*} of the stationary marked point processes X_{L,C^e} and X_{L,C^g} introduced in Section 2.3. Note that the realizations of X_{L,C^e} can be constructed from the corresponding realizations of X_L and X_{H,S^e}, where $X_{H,S^e} = \{(X_{H,n}, S^{e}_{H,n})\}$. Thus, instead of X_{L,C^e}, we can consider the vector $Y = (X_L, X_{H,S^e})$ and the Palm distribution $\mathbb{P}^*_{X_L}$ with respect to X_L, which has been introduced in (2).

Let $(X_L^*, \widetilde{X}_{H,S^e})$ be distributed according to $\mathbb{P}^*_{X_L}$, where we use the notation

$$\widetilde{X}_{H,S^e} = \{(\widetilde{X}_{H,n}, \widetilde{S}^e_{H,n})\} \qquad \text{and} \qquad \widetilde{G}_\gamma = \bigcup_{n\geq 1} \left(\widetilde{S}^e_{H,n} + \widetilde{X}_{H,n}\right). \tag{7}$$

Let $\widetilde{X}^e_{H,o}$ denote the closest point (in the Euclidean sense) of $\{\widetilde{X}_{H,n}\}$ to the origin o. Then, the typical shortest–path length $C^{e*} = C^{e*}(\gamma, \lambda_\ell, \lambda'_\ell)$ can be given by

$$C^{e*} = c(\widetilde{X}^e_{H,o}), \tag{8}$$

where $c(\widetilde{X}^e_{H,o})$ denotes the length of the shortest path from o to $\widetilde{X}^e_{H,o}$ along the edges of $\widetilde{G}_\gamma$. These remarks are of general nature and remain true if we pass from the Euclidean to the graph metric. In particular, the typical shortest–path length $C^{g*} = C^{g*}(\gamma, \lambda_\ell, \lambda'_\ell)$ can be given by

$$C^{g*} = c(\widetilde{X}^g_{H,o}), \tag{9}$$

where $c(\widetilde{X}^g_{H,o})$ denotes the length of the shortest path from o to the g-closest point $\widetilde{X}^g_{H,o}$ of $\{\widetilde{X}_{H,n}\}$ along the edges of $\widetilde{G}_\gamma$.

Consider the stationary (marked) Cox processes $X_{H,S^e} = \{(X_{H,n}, S^e_{H,n})\}$ and $X_{H,S^g} = \{(X_{H,n}, S^g_{H,n})\}$ introduced in Section 2.3. Their Palm mark distributions can be used to get the following representation formulae for the distributions of C^{e*} and C^{g*}.

Lemma 4.1. *Let $h : \mathbb{R} \to [0, \infty)$ be a Borel–measurable function which is non–negative. Then,*

$$\mathbb{E}h(C^{e*}) = \lambda_\ell \mathbb{E} \int_{S^{e*}_H} h(c(y))\nu_1(dy) \tag{10}$$

and

$$\mathbb{E}h(C^{g*}) = \lambda_\ell \mathbb{E} \int_{S^{g*}_H} h(c(y))\nu_1(dy), \tag{11}$$

where S^{e*}_H and S^{g*}_H denote the typical marks of X_{H,S^e} and X_{H,S^g}, respectively, and $c(y)$ is the shortest path length from y to o along the edges of S^{e*}_H resp. S^{g*}_H.

Proof. Having in mind that C^{*e} and S^{*e}_H are the typical marks of the jointly stationary marked point processes $X_{L,C}$ and X_{H,S^e}, formula (10) easily follows from Neveu's exchange formula, see [12]. Formula (11) is obtained in the same way. □

From (10) and (11), it can be seen that the distributions of C^{e*} and C^{g*} do not depend on λ'_ℓ. Furthermore, the following result is true.

Proposition 4.2. *For any $(\gamma, \lambda_\ell, \lambda'_\ell) \in [0, \infty)^3$, it holds that $C^{g*} \leq_{\rm st} C^{e*}$, i.e.,*

$$\mathbb{P}(C^{g*} \leq t) \geq \mathbb{P}(C^{e*} \leq t) \qquad \text{for all } t > 0. \tag{12}$$

Proof. By the definitions of e-closeness and g-closeness, i.e., by the definitions of the random variables $\widetilde{X}^e_{H,o}$ and $\widetilde{X}^g_{H,o}$ introduced in Section 2.3 we get that $c(\widetilde{X}^g_{H,o}) \leq c(\widetilde{X}^e_{H,o})$ with $\mathbb{P}^*_{X_L}$-probability 1. Thus, (12) immediately follows from (8) and (9). □

In order to show that $C^{g*} = C^{g*}(\gamma, \lambda_\ell, \lambda'_\ell)$ stochastically decreases in λ_ℓ, the following auxiliary result is useful.

Lemma 4.3. *The point process $\widetilde{X}_H = \{\widetilde{X}_{H,n}\}$ introduced in (7) is a (non-stationary) Cox process on $\widetilde{G}_\gamma$ with linear intensity λ_ℓ, i.e., for each $B \in \mathcal{B}^2$ it holds that $\mathbb{E}\widetilde{X}_H(B) = \lambda_\ell\, \mathbb{E}\nu_1(B \cap \widetilde{G}_\gamma)$.*

Proof. The assertion immediately follows from Slivnyak's theorem for stationary Cox processes, see e.g. [15], p. 156. □

Proposition 4.4. *For any fixed $(\gamma, \lambda'_\ell) \in [0,\infty)^2$, the typical shortest–path length $C^{g*} = C^{g*}(\gamma, \lambda_\ell, \lambda'_\ell)$ stochastically decreases in λ_ℓ, i.e.,*

$$C^{g*}(\gamma, \lambda_{\ell,2}, \lambda'_\ell) \leq_{\mathrm{st}} C^{g*}(\gamma, \lambda_{\ell,1}, \lambda'_\ell) \qquad \textit{if } \lambda_{\ell,1} \leq \lambda_{\ell,2}. \tag{13}$$

Proof. For any $\lambda_\ell > 0$, we consider the point process $\widetilde{X}_H = \widetilde{X}_{H,\lambda_\ell}$ introduced in (7), which is a Cox process with linear intensity λ_ℓ according to Lemma 4.3. Given $\widetilde{G}_\gamma$, this means that $\widetilde{X}_{H,\lambda_\ell}$ is a homogeneous Poisson process on $\widetilde{G}_\gamma$ with intensity λ_ℓ. Consequently, due to the well–known invariance property of Poisson processes with respect to convolution, we get that for any $0 < \lambda_{\ell,1} \leq \lambda_{\ell,2}$:

$$\widetilde{X}_{H,\lambda_{\ell,2}} \stackrel{\mathrm{d}}{=} \widetilde{X}_{H,\lambda_{\ell,1}} + \widetilde{X}_{H,\lambda_{\ell,2}-\lambda_{\ell,1}}\,,$$

where the Cox processes $\widetilde{X}_{H,\lambda_{\ell,1}}$ and $\widetilde{X}_{H,\lambda_{\ell,2}}$ are assumed to be conditionally independent given $\widetilde{G}_\gamma$. For the length $c(\widetilde{X}^g_{H,o}(\lambda_\ell))$ of the shortest path from o to the g-closest point $\widetilde{X}^g_{H,0} = \widetilde{X}^g_{H,0}(\lambda_\ell)$ of $\widetilde{X}_{H,\lambda\ell}$ along the edges of $\widetilde{G}_\gamma$, this implies that

$$c(\widetilde{X}^g_{H,o}(\lambda_{\ell,2})) \leq_{\mathrm{st}} c(\widetilde{X}^g_{H,o}(\lambda_{\ell,1})) \tag{14}$$

for any $\lambda_{\ell,1}, \lambda_{\ell,2} > 0$ with $\lambda_{\ell,1} \leq \lambda_{\ell,2}$. Using (9), this completes the proof. □

Unfortunately, the argument which leads to (14) seems not to work for e-closeness. However, we conjecture that also $C^{e*} = C^{e*}(\gamma, \lambda_\ell, \lambda'_\ell)$ stochastically decreases in λ_ℓ, as indicated by the numerical results stated in Section 6.

5 Limit theorems

The study of limit cases of unboundedly sparse or dense edge sets is an important matter for modeling of realistic networks. The link between theoretical work and applications are the parametric functions fitted to empirical distributions of typical path lengths (see Section 6.2). Those formulas are embedded in specialized software and are expected to produce results for all possible densities of edges sets. The computation of empirical densities for shortest paths can be done for a large range around moderate density values, but is not possible for very high or low ones. If theoretical results for limit cases exist, one can fill the gap in a way based on sound considerations. We investigate the asymptotic behavior of the distributions of C^{e*} and C^{g*} for two different cases: $\kappa = \gamma/\lambda_\ell \to 0$ and $\kappa \to \infty$. First observe the simple scaling relations $C^{e*}(\gamma, \lambda_\ell) \stackrel{d}{=} aC^{e*}(a\gamma, a\lambda_\ell)$ and $C^{g*}(\gamma, \lambda_\ell) \stackrel{d}{=} aC^{g*}(a\gamma, a\lambda_\ell)$. Therefore it suffices to understand the limit behaviour as $\kappa \to 0$ under the additional assumption of fixed λ_ℓ. Similarly, in the case $\kappa \to \infty$ we may assume that $\gamma \to \infty$ and $\lambda_\ell \to 0$ in a way that $\lambda_\ell \gamma$ remains fixed. For $\kappa \to 0$, we show in Theorem 5.1 that the distributions of C^{e*} and C^{g*} converge to an exponential distribution, where no additional assumptions on the underlying random geometric graph G are needed. Furthermore, for $\kappa \to \infty$, we show in Theorem 5.2 that the distributions of C^{e*} and C^{g*} converge to a Weibull distribution provided that G is isotropic and mixing satisfying $G' = G$ and $\mathbb{E}\nu_1^2(\partial \Xi^*) < \infty$, where Ξ^* denotes the typical cell of G.

We denote by $\mathrm{Exp}(\delta)$ an exponential distribution with expectation δ^{-1}. Furthermore, $\mathrm{Wei}(a, b)$ denotes the Weibull distribution with scale parameter $a > 0$ and shape parameter $b > 0$.

5.1 Asymptotic exponential distribution for $\kappa \to 0$

First we regard the case that $\kappa = \gamma/\lambda_\ell \to 0$ with λ_ℓ fixed, i.e., $\gamma \to 0$.

Theorem 5.1. *Let G be an arbitrary random geometric graph which is stationary and connected. Then, for any fixed $\lambda_\ell > 0$, it holds that*

$$C^{e*}(\gamma, \lambda_\ell) \stackrel{\mathrm{d}}{\to} Z \qquad \text{and} \qquad C^{g*}(\gamma, \lambda_\ell) \stackrel{\mathrm{d}}{\to} Z \tag{15}$$

if $\gamma \to 0$, where $\stackrel{\mathrm{d}}{\to}$ denotes convergence in distribution and $Z \sim \mathrm{Exp}(2\lambda_\ell)$.

Proof. We can use similar arguments as in [17], where the asymptotic behavior of C^{e*} has been considered for the special case that G is the edge set of a stationary tessellation with convex cells. Namely, it holds that $\lim_{\gamma\to 0} R_\gamma = \infty$ with $\mathbb{P}^*_{X_L}$–probability 1, where $R_\gamma = \max\{r > 0 : B(o, r) \cap \widetilde{S}^o_\gamma = B(o, r) \cap \widetilde{G}_\gamma\}$ and $\widetilde{S}^o_\gamma$ denotes the segment of $\widetilde{G}_\gamma$ containing the origin. This implies that

$$\lim_{\gamma\to 0} \mathbb{P}(C^{*g} \le x \mid \widetilde{X}^g_{H,o} \in B(o, R_\gamma)) = \lim_{\gamma\to 0} \mathbb{P}(C^{*e} \le x \mid \widetilde{X}^e_{H,o} \in B(o, R_\gamma))\,.$$

Thus, in the same way as done in [17] for C^{e*} and G being the edge set of a stationary tessellation with convex cells, it can be shown that C^{e*} and C^{g*} converge in distribution to the random distance from o to the nearest point of a stationary Poisson process on $\mathbb{R}$ with intensity λ_ℓ, which is $\mathrm{Exp}(2\lambda_\ell)$-distributed. □

5.2 Asymptotic Weibull distribution for $\kappa \to \infty$

In this section we additionally assume that the stationary edge set G is isotropic and mixing. Furthermore, we assume that G is fully connected, does not possess dead ends, i.e. $G' = G$, and

$$\mathbb{E}\,\nu_1^2(\partial \Xi^*) < \infty\,, \tag{16}$$

where $\nu_1(\partial \Xi^*)$ denotes the circumference of the typical cell Ξ^* of G. Then, it is not difficult to see that the proof of Theorem 3.2 in [17] regarding the asymptotic behavior of $C^{e*}(\gamma, \lambda_\ell)$ as $\kappa \to \infty$ remains true in the present case of a random tessellation G with not necessarily convex cells. Our goal is to show that the result derived in Theorem 3.2 of [17] also holds if we pass from $C^{e*}(\gamma, \lambda_\ell)$ to $C^{g*}(\gamma, \lambda_\ell)$.

Theorem 5.2. *If $\gamma \to \infty$ and $\lambda_\ell \to 0$ with $\lambda_\ell \gamma = \lambda$ fixed, then there exists a constant $\xi \geq 1$ such that*

$$C^{e*}(\gamma, \lambda_\ell) \xrightarrow{\text{d}} \xi Z \quad \text{and} \quad C^{g*}(\gamma, \lambda_\ell) \xrightarrow{\text{d}} \xi Z\,, \tag{17}$$

where $\xi Z \sim \text{Wei}(\lambda\pi/\xi^2, 2)$.

In order to show Theorem 5.2, let us recall the idea behind the proof of Theorem 3.2 in [17]. By an argument based on Kingman's subadditive ergodic theorem one can show that there exists $\xi \geq 1$ such that $C^{e*} - \xi|\widetilde{X}^e_{H,o}| \xrightarrow{\text{P}} 0$, where $\xrightarrow{\text{P}}$ denotes convergence in probability. Since $|\widetilde{X}^e_{H,o}| \xrightarrow{\text{d}} \text{Wei}(\lambda\pi, 2)$, an application of Slutsky's lemma (see [6, Chapter 2, Exc. 2.10]) yields $C^{e*} \xrightarrow{\text{d}} \text{Wei}(\lambda\pi/\xi^2, 2)$. Moreover, this strategy proves to be flexible enough to handle the current situation with respect to C^{g*} as well.

To show that $C^{g*} \xrightarrow{\text{d}} \text{Wei}(\lambda\pi/\xi^2, 2)$ holds, we begin by proving an analogue of Lemma 4.4 in [17].

Lemma 5.3. *Let $\gamma \to \infty$ and $\lambda_\ell \to 0$ such that $\lambda_\ell\gamma = \lambda$ is fixed. Then there is a constant $\xi \geq 1$ such that $C^{g*}(\gamma, \lambda_\ell) - \xi|\widetilde{X}^g_{H,o}| \xrightarrow{\text{P}} 0$.*

Proof. Let $\varepsilon, \delta > 0$ be arbitrary. We want to prove that there exists a $\xi \geq 1$ (which in fact coincides with the ξ in Lemma 4.4 of [17]) such that for all γ sufficiently large we have $\mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon\big) \leq \delta$. Using

$$\begin{aligned}\mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon\big) &= \mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon, |\widetilde{X}^g_{H,o}| \leq r\big) \\ &\quad + \mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon, |\widetilde{X}^g_{H,o}| > r\big),\end{aligned}$$

it suffices to show that there exists $r > 0$ such that for all sufficiently large γ we have

$$\mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon, |\widetilde{X}^g_{H,o}| \leq r\big) < \delta/2 \quad \text{and} \quad \mathbb{P}\big(|\widetilde{X}^g_{H,o}| > r\big) < \delta/2\,.$$

Using the elementary inequalities $|\widetilde{X}^g_{H,o}| \le c(\widetilde{X}^g_{H,o}) \le c(\widetilde{X}^e_{H,o})$ and Lemma 4.2 of [17], i.e. $|\widetilde{X}^e_{H,o}| \xrightarrow{d} \mathrm{Wei}(\lambda\pi, 2)$, we can choose $r > 0$ such that

$$\begin{aligned}\mathbb{P}\big(|\widetilde{X}^g_{H,o}| > r\big) &\le \mathbb{P}\big(c(\widetilde{X}^e_{H,o}) > r\big)\\ &\le \mathbb{P}\big(\xi|\widetilde{X}^e_{H,o}| > r/2\big) + \mathbb{P}\big(\big|\xi|\widetilde{X}^e_{H,o}| - c(\widetilde{X}^e_{H,o})\big| > r/2\big)\\ &\le \frac{\delta}{4} + \frac{\delta}{4}\end{aligned}$$

for all $\gamma > 0$ sufficiently large. To prove the remaining estimate, we denote by $\widetilde{N} = \widetilde{X}_H(B(o,r))$ be the number of points of $\widetilde{X}_H$ in the ball $B(o,r)$. Using this notation, we may write

$$\begin{aligned}&\mathbb{P}\big(\big|C^{g*} - \xi|\widetilde{X}^g_{H,o}|\big| > \varepsilon, |\widetilde{X}^g_{H,o}| \le r\big)\\ &\le \mathbb{E}\Big(\sum_{k=1}^{\infty} \mathbb{P}(\widetilde{N} = k \mid \widetilde{G}_\gamma) \cdot \mathbb{E}\,\big(\max_{i=1,\dots,k} \big|c(Y_i) - \xi|Y_i|\big| > \varepsilon \mid \widetilde{G}_\gamma, \widetilde{N} = k)\Big)\,,\end{aligned}$$

where the random vectors $Y_1, \dots, Y_k$ are conditionally independent and uniformly distributed on $\widetilde{G}_\gamma \cap B(o,r)$ given $\widetilde{G}_\gamma$ and $\widetilde{N}$. Then, in the next steps, copying the corresponding part of the proof of Lemma 4.4 in [17] verbatim yields the assertion. □

As a second ingredient in the proof of $C^{g*} \xrightarrow{\mathrm{d}} \mathrm{Wei}(\lambda\pi/\xi^2, 2)$, we still need another auxiliary result.

Lemma 5.4. *Let $\gamma \to \infty$ and $\lambda_\ell \to 0$ such that $\lambda_\ell\gamma = \lambda$ is fixed. Then,*

$$|\widetilde{X}^g_{H,o}| - |\widetilde{X}^e_{H,o}| \xrightarrow{\mathrm{P}} 0 \tag{18}$$

and consequently,

$$|\widetilde{X}^g_{H,o}| \xrightarrow{\mathrm{d}} \mathrm{Wei}(\lambda\pi, 2)\,. \tag{19}$$

Proof. Observe that $|\widetilde{X}^g_{H,o}| - |\widetilde{X}^e_{H,o}| \ge 0$ holds by definition. Furthermore, for any $\xi > 0$ we have

$$\begin{aligned}&\mathbb{P}\big(\xi|\widetilde{X}^g_{H,o}| - \xi|\widetilde{X}^e_{H,o}| > \varepsilon\big)\\ &= \mathbb{P}\big((\xi|\widetilde{X}^g_{H,o}| - c(\widetilde{X}^g_{H,o})) + (c(\widetilde{X}^g_{H,o}) - c(\widetilde{X}^e_{H,o})) + (c(\widetilde{X}^e_{H,o}) - \xi|\widetilde{X}^e_{H,o}|) > \varepsilon\big)\\ &\le \mathbb{P}\big(\big|\xi|\widetilde{X}^g_{H,o}| - c(\widetilde{X}^g_{H,o})\big| > \varepsilon/2\big) + \mathbb{P}\big(\big|c(\widetilde{X}^e_{H,o}) - \xi|\widetilde{X}^e_{H,o}|\big| > \varepsilon/2\big)\,,\end{aligned}$$

where in the last inequality we used that $c(\widetilde{X}^g_{H,o}) - c(\widetilde{X}^e_{H,o}) \le 0$. But, by Lemma 4.4 of [17] and by Lemma 5.3, we know that both summands of the latter bound converge to 0 as $\gamma \to \infty$. This proves (18) and, applying Slutsky's lemma, (19) follows by taking into account that $|\widetilde{X}^e_{H,o}| \xrightarrow{\mathrm{d}} \mathrm{Wei}(\lambda\pi, 2)$. □

Note that the convergence $C^{g*}(\gamma, \lambda_\ell) \xrightarrow{\mathrm{d}} \xi Z$ stated in Theorem 5.2 is now an immediate consequence of Lemmas 5.3 and 5.4, because

$$c(\widetilde{X}^g_{H,o}) = \xi|\widetilde{X}^g_{H,o}| + \big(c(\widetilde{X}^g_{H,o}) - \xi|\widetilde{X}^g_{H,o}|\big) \xrightarrow{\mathrm{d}} \xi Z \sim \mathrm{Wei}(\lambda\pi/\xi^2, 2)\,.$$

5.3 Checking the conditions of Theorem 5.2

We now discuss some examples of G for which the conditions of Theorem 5.2 are satisfied. If G is the edge set of a PLT, PVT, and PDT, respectively, then G is isotropic by definition and it is well known that G is mixing, see Chapter 10.5 in [14]. Furthermore, it is not difficult to show in all these cases that the integrability condition (16) is fulfilled, see [17].

Moreover, it turns out that the conditions of Theorem 5.2 are fulfilled if G is the edge set of an PVAT introduced in Section 3.2. Then, by definition, G is stationary and isotropic. In order to show that G is mixing, we can apply an extended version of the arguments which have been used in Theorem 10.5.1 of [14] for (non–aggregated) PVT.

Lemma 5.5. *Let $T^{(1)} = \{\Xi_n^{(1)}\}$ and $T^{(2)} = \{\Xi_n^{(2)}\}$ be independent Poisson–Voronoi tessellations. Then the edge set $G = \bigcup_{n\geq 1} \partial \Xi_n$ of the* PVAT *$T = \{\Xi_n\}$ introduced in* (3) *is mixing.*

Proof. To begin with, we recall the basic ideas of the proof of Theorem 10.5.1 in [14], which have been developed there to show that any (non–aggregate) PVT $T = \{\Xi_n\}$ is mixing, where $\Xi_n = \{x \in \mathbb{R}^2 : |X_n - x| \leq |X_i - x| \text{ for any } i \neq n\}$ and $X = \{X_n\}$ is a homogeneous Poisson process. That is, for any fixed $\varepsilon > 0$ and bounded $B_1, B_2 \in \mathcal{B}^2$ it holds that

$$\left|\mathbb{P}\big(G \cap B_1 = \emptyset,\, G \cap (B_2 + x) = \emptyset\big) - \mathbb{P}(G \cap B_1 = \emptyset) P(G \cap (B_2 + x) = \emptyset)\right| < 6\varepsilon \tag{20}$$

for all $x \in \mathbb{R}^2$ such that $|x|$ is sufficiently large, where $G = \bigcup_{n\geq 1} \partial \Xi_n$ denotes the edge set of T. In order to prove (20) the following truncation technique has been used in [14]. For $r > 0$ and $x \in \mathbb{R}^2$, let G_x denote the edge set of the Voronoi tessellation induced by the point process $X \cap B(x, 15r)$. If $|x| > 30r$, then the random sets G_o and G_x are independent. This implies that

$$\mathbb{P}(G_o \cap B_1 = \emptyset, G_x \cap (B_2 + x) = \emptyset) = \mathbb{P}(G_o \cap B_1 = \emptyset)\mathbb{P}(G_x \cap (B_2 + x) = \emptyset).$$

The next step in the proof of Theorem 10.5.1 given in [14], which yields (20), is to show that one can choose $r > 0$ such that

$$\begin{aligned}
&\left|\mathbb{P}(G_o \cap B_1 = \emptyset, G_x \cap (B_2 + x) = \emptyset) - \mathbb{P}(G \cap B_1 = \emptyset, G \cap (B_2 + x) = \emptyset)\right| < 2\varepsilon \\
&\left|\mathbb{P}(G_o \cap B_1 = \emptyset) - \mathbb{P}(G \cap B_1 = \emptyset)\right| < 2\varepsilon \\
&\left|\mathbb{P}(G_x \cap (B_2 + x) = \emptyset) - \mathbb{P}(G \cap (B_2 + x) = \emptyset)\right| < 2\varepsilon
\end{aligned} \tag{21}$$

holds for all $x \in \mathbb{R}^2$ with $|x|$ sufficiently large. To achieve this goal, one first introduces the following family of events (parametrized by $y \in \mathbb{R}^2$ and $r > 0$)

$$E_r^y = \{\omega \in \Omega : \Xi_n(\omega) \cap B(y, r) \neq \emptyset, \Xi_n(\omega) \not\subset B(y, 5r) \text{ for some } n \geq 1\}.$$

It is not difficult to see that for $r > 0$ sufficiently large, we have $\mathbb{P}(E_r^y) = \mathbb{P}(E_r^o) < \varepsilon$ for all $y \in \mathbb{R}^2$. This yields $\mathbb{P}(F^{x_1,x_2}) \geq 1 - 2\varepsilon$ for any $x_1, x_2 \in \mathbb{R}^2$, where $F_r^{x_1,x_2} = (E_r^{x_1} \cup E_r^{x_2})^c$. The benefit of introducing the events E_r^y and

$F_r^{x_1,x_2}$ is as follows. First choose $r > 0$ large enough such that $B_1, B_2 \subset B(o, r)$. Now suppose that $\omega \in F_r^{x_1,x_2}$ and $\Xi_n(\omega) \cap (B_i + x_i) \neq \emptyset$ for some $n \geq 1$ and $i \in \{1, 2\}$. Then $\Xi_n(\omega) \subset B(x_i, 5r)$. Using elementary geometry, it is easy to see that this implies that the Voronoi cell $\Xi_n(\omega)$ is completely determined by $X \cap B(x_i, 15r)$. Thus, $G \cap (B_i + x_i) = \emptyset$ holds if and only if $G_{x_i} \cap (B_i + x_i) = \emptyset$ implying the inequalities in (21). The basic ideas of this approach may also be used in the case of PVAT. Denote by $X^{(1)}$ and $X^{(2)}$ independent Poisson processes inducing the PVT $T^{(1)} = \{\Xi_n^{(1)}\}$ and $T^{(2)} = \{\Xi_n^{(2)}\}$, respectively. For $y \in \mathbb{R}^2$ and $r > 0$, let

$$
\begin{aligned}
E_r^y = &\{\omega \in \Omega : \Xi_n^{(2)}(\omega) \cap B(y, r) \neq \emptyset, \Xi_n^{(2)}(\omega) \not\subset B(y, 5r) \text{ for some } n \geq 1\} \\
&\cup \{\omega \in \Omega : \Xi_n^{(1)}(\omega) \cap B(y, 5r) \neq \emptyset, \Xi_n^{(1)}(\omega) \not\subset B(y, 25r) \text{ for some } n \geq 1\} \\
&\cup \{\omega \in \Omega : \Xi_n^{(2)}(\omega) \cap B(y, 25r) \neq \emptyset, \Xi_n^{(2)}(\omega) \not\subset B(y, 125r) \text{ for some } n \geq 1\} .
\end{aligned}
$$

Then, like in the case of a (non–aggregate) PVT mentioned above, it is easy to see that for $r > 0$ sufficiently large, we have $\mathbb{P}(E_r^y) = \mathbb{P}(E_r^o) < \varepsilon$ for all $y \in \mathbb{R}^2$ and $\mathbb{P}(F^{x_1,x_2}) \geq 1 - 2\varepsilon$ for any $x_1, x_2 \in \mathbb{R}^2$, where $F_r^{x_1,x_2} = (E_r^{x_1} \cup E_r^{x_2})^c$. Furthermore, if $\omega \in F_r^{x_1,x_2}$, then $\Xi_n(\omega) \cap (B_i + x_i) \neq \emptyset$ for some $n \geq 1$ and $i \in \{1, 2\}$ implies that $\Xi_n(\omega) \subset B(x_i, 125r)$, where $T = \{\Xi_n\}$ is the PVAT induced by $X^{(1)}$ and $X^{(2)}$. Thus, in this case, the cell Ξ_n is completely determined by $X^{(1)} \cap B(x_i, 375r)$ and $X^{(2)} \cap B(x_i, 375r)$ and the proof can be finished in the same way as indicated above for (non–aggregate) PVT. □

Finally, we show that the integrability condition (16) is fulfilled if G is the edge set of an PVAT.

Lemma 5.6. *Let $T^{(1)} = \{\Xi_n^{(1)}\}$ and $T^{(2)} = \{\Xi_n^{(2)}\}$ be Voronoi tessellations induced by the stationary and independent point processes $X^{(1)} = \{X_n^{(1)}\}$ and $X^{(2)} = \{X_n^{(2)}\}$ with intensities $\lambda^{(1)}$ and $\lambda^{(2)}$. Assume that $\mathbb{E}(\nu_2^4(\Xi^{(1),*})) < \infty$ and $\mathbb{E}(\nu_1^4(\partial\Xi^{(2),*})) < \infty$ holds, where $\Xi^{(1),*}$ and $\Xi^{(2),*}$ denotes the typical cell of $T^{(1)}$ and $T^{(2)}$, respectively. If $X^{(2)} = \{X_n^{(2)}\}$ is a homogeneous Poisson process, then $\mathbb{E}(\nu_1^2(\partial\Xi^*)) < \infty$ holds, where Ξ^* is the typical cell of the aggregate tessellation $T = \{\Xi_n\}$ induced by $X^{(1)}$ and $X^{(2)}$.*

Proof. Consider the marked point processes $Y^{(1)} = \{(X_n^{(1)}, \Xi_n^{(1)} - X_n^{(1)})\}$ and $Y^{(2)} = \{(X_n^{(2)}, \Xi_n^{(2)} - X_n^{(2)})\}$. Let $\mathbb{P}_1^o$ denote the Palm mark distribution of $Y^{(1)}$, see (1), and let $\mathbb{C}_2$ be the Campbell measure of $Y^{(2)}$, i.e.,

$$
\mathbb{C}_2(B \times E \times A) = \mathbb{E}\left(\#\{n : X_n^{(2)} \in B, \Xi_n^{(2)} - X_n^{(2)} \in E\} \, \mathbb{1}_A(X^{(2)})\right),
$$

where $B \in \mathcal{B}^2$, $E \in \mathcal{B}_{\mathcal{F}}$, and $A \in \mathcal{N}$. Then, using (1) and (3), we get that

$$
\begin{aligned}
\mathbb{E}(\nu_1^2(\partial\Xi^*) &\leq \frac{1}{\lambda^{(1)}} \mathbb{E}\Big(\sum_{n:\, X_n^{(1)} \in [0,1]^2} \Big(\sum_{i:\, X_i^{(2)} \in \Xi_n^{(1)}} \nu_1(\partial\Xi_i^{(2)}) \Big)^2 \Big) \\
&\leq \frac{1}{\lambda^{(1)}} \mathbb{E}\Big(\sum_{n:\, X_n^{(1)} \in [0,1]^2} X^{(2)}(\Xi_n^{(1)}) \sum_{i:\, X_i^{(2)} \in \Xi_n^{(1)}} \nu_1^2(\partial\Xi_i^{(2)}) \Big) .
\end{aligned}
$$

Thus, using Campbell's theorem for stationary marked point processes (see e.g. [5,15]), this gives

$$\begin{aligned}\mathbb{E}(\nu_1^2(\partial\Xi^*)) &\le \int_{[0,1]^2}\int_{\mathcal{F}}\int_{\mathbb{R}^2\times\mathcal{F}\times N} \mathbb{1}_{m_1+x}(y)\varphi(m_1+x)\nu_1^2(\partial m_2)\,\mathbb{C}_2(dy,dm_2,d\varphi)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{P}_1^o(dm_1)\,dx\\ &\le \int_{[0,1]^2}\int_{\mathcal{F}}\Big(\int_{\mathbb{R}^2\times\mathcal{F}\times N} \mathbb{1}_{m_1+x}(y)\varphi^2(m_1+x)\,\mathbb{C}_2(dy,dm_2,d\varphi)\Big)^{1/2}\\ &\quad\times\Big(\int_{\mathbb{R}^2\times\mathcal{F}\times N} \mathbb{1}_{m_1+x}(y)\nu_1^4(\partial m_2)\,\mathbb{C}_2(dy,dm_2,d\varphi)\Big)^{1/2}\mathbb{P}_1^o(dm_1)\,dx\,.\end{aligned}$$

Furthermore, by the definition of the typical cell $\Xi^{(2),*}$, we have

$$\int_{\mathbb{R}^2\times\mathcal{F}\times N} \mathbb{1}_{m_1+x}(y)\nu_1^4(\partial m_2)\,\mathbb{C}_2(dy,dm_2,d\varphi) = \lambda^{(2)}\nu_2(m_1)\,\mathbb{E}\nu_1^4(\partial\Xi^{(2),*})\,,$$

and applying Slivnyak's theorem (see [5,15]) to the homogeneous Poisson process $X^{(2)}$ gives that

$$\begin{aligned}&\int_{\mathbb{R}^2\times\mathcal{F}\times N} \mathbb{1}_{m_1+x}(y)\varphi^2(m_1+x)\,\mathbb{C}_2(dy,dm_2,d\varphi)\\ &\qquad\qquad = \big(\lambda^{(2)}\big)^2\nu_2^3(m_1)+3\lambda^{(2)}\nu_2^2(m_1)+\nu_2(m_1)\,.\end{aligned}$$

Thus, we get that

$$\begin{aligned}\mathbb{E}(\nu_1^2(\partial\Xi^*)) &\le \Big(\lambda^{(2)}\,\mathbb{E}\nu_1^4(\partial\Xi^{(2),*})\Big)^{1/2}\int_{\mathcal{F}}\Big(\big(\lambda^{(2)}\big)^2\nu_2^4(m_1)+3\lambda^{(2)}\nu_3^2(m_1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad + \nu_2^2(m_1)\Big)^{1/2}\mathbb{P}_1^o(dm_1)\\ &\le \Big(\lambda^{(2)}\,\mathbb{E}\nu_1^4(\partial\Xi^{(2),*})\Big)^{1/2}\Big(\mathbb{E}(\nu_2^2(\Xi^{(1),*}))+3\lambda^{(2)}\mathbb{E}(\nu_2^3(\Xi^{(1),*}))\\ &\qquad\qquad\qquad\qquad\qquad\qquad + (\lambda^{(2)})^2\mathbb{E}(\nu_2^4(\Xi^{(1),*}))\Big)^{1/2},\end{aligned}$$

where the latter bound is finite, because we assumed that $\mathbb{E}(\nu_2^4(\Xi^{(1),*}))<\infty$ and $\mathbb{E}(\nu_1^4(\partial\Xi^{(2),*}))<\infty$. □

Corollary 5.7. *If both $X^{(1)}$ and $X^{(2)}$ are homogeneous Poisson processes, then $\mathbb{E}(\nu_1^2(\partial\Xi^*))<\infty$ holds, where Ξ^* is the typical cell of the* PVAT $T=\{\Xi_n\}$ *induced by $X^{(1)}$ and $X^{(2)}$.*

Proof. Let $B\in\mathcal{B}^2$ be bounded and convex, and denote by $R(B)$ the radius of the smallest ball containing B. By the convexity of B it holds that $\nu_1(\partial B)\le 2\pi R(B)$ and $\nu_2(B)\le\pi R^2(B)$. Thus, by the result of Lemma 5.6, it is sufficient to show that all moments of $R(\Xi^{(i),*})$ are finite, where $\Xi^{(i),*}$ denotes the typical cell of the PVT $T^{(i)}$ for $i\in\{1,2\}$. But this follows from a result derived in [7] (see also [4]), where it is shown that $R(\Xi^{(i),*})$ has exponential tails. □

6 Numerical results

6.1 Simulation–based density estimators

The aim of this section is to consider simulation–based estimators for the densities of the typical shortest–path lengths C^{e*} and C^{g*}. In particular, we construct an estimator for the density $f_{C^{g*}}(x)$ of C^{g*} based on n independent and identically distributed samples of the typical segment system S_H^{g*}, similar to the estimator $\hat{f}_{C^{e*}}(x;n)$ for the density of C^{e*} introduced in [18].

To achieve this goal it is useful to decompose the typical segment system S_H^{g*}, i.e., we subdivide S_H^{g*} into N parts, its segments $e_1, \ldots, e_I$, and denote by A_i and B_i, $i \in \{1, ..., I\}$, the endpoints of these segments (see Figure 6), such that it holds:

- $S_H^{*,g} = \bigcup_{i=1}^{I} e_i$
- $\nu_1(e_i \cap e_j) = 0$ for $i \neq j$ and
- $c(A_i) < c(B_i) = c(A_i) + \nu_1(e_i)$.

Note that it can sometimes happen that so-called *distance peaks* occur. A point x_{dp} is called distance peak if there exist two different shortest paths from o to x_{dp}. Therefore, some segments are subdivided into two parts at x_{dp} (see Figure 6).

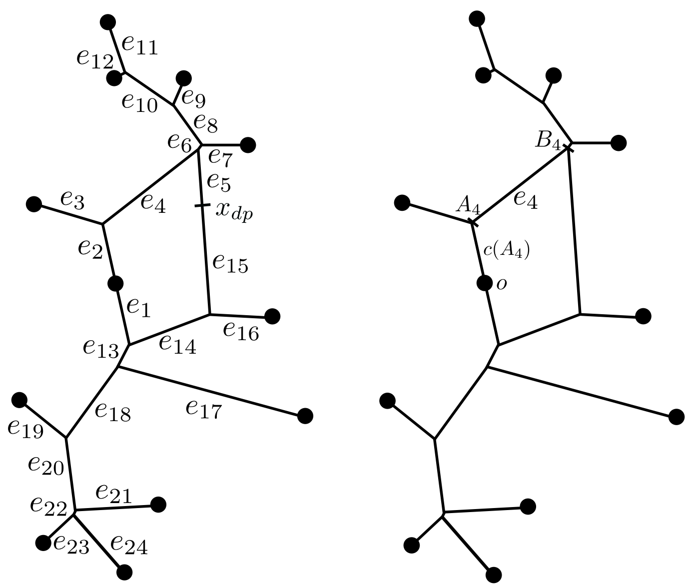

Fig. 6 Typical segment system and its subdivision

In the same way as done in Theorem 1 of [18] for the density of C^{e*}, the following result is obtained.

Proposition 6.1. *The typical shortest–path length C^{g*} is an absolutely continuous random variable and its density $f_{C^{g*}}(x)$ is given by*

$$f_{C^{g*}}(x) = \begin{cases} \lambda_\ell \mathbb{E}(\sum_{i=1}^{I} \mathbb{1}_{[c(A_i),c(B_i))}(x)) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Proof. For $B \in \mathcal{B}$ and $h(x) = \mathbb{1}_B(x)$, we get from (11) that

$$\begin{aligned} \mathbb{P}_{C^{g*}}(B) &= \lambda_\ell \mathbb{E} \sum_{i=1}^{I} \int_{c(A_i)}^{c(B_i)} \mathbb{1}_B(x) dx \\ &= \int_B \lambda_\ell \mathbb{E} \sum_{i=1}^{I} \mathbb{1}_{[c(A_i),c(B_i))}(x) dx \,. \end{aligned}$$

This is equivalent with (22). □

Let $S^{g*}_{H,1}$,...,$S^{g*}_{H,n}$ be n independent and identically distributed samples of the typical segment system S^{g*}_H and consider the segments $e^{(j)}_1, ..., e^{(j)}_{I_j}$ and the shortest–path lengths $c(A^{(j)}_1), c(B^{(j)}_1), ..., c(A^{(j)}_{I_j}), c(B^{(j)}_{I_j})$ from their endpoints to o; $j = 1, \ldots, n$. Then, in view of (22), a suitable estimator $\hat{f}_{C^{g*}}(x; n)$ for the density $f_{C^{g*}}(x)$ of C^{g*} can be defined by

$$\hat{f}_{C^{g*}}(x; n) = \lambda_\ell \, \frac{1}{n} \, \sum_{j=1}^{n} \sum_{i=1}^{I_j} \mathbb{1}_{[c(A^{(j)}_i),c(B^{(j)}_i))}(x) \,.$$

Note the following statistical properties of $\hat{f}_{C^{g*}}(x; n)$ which can be easily proven, see also [18]:

1) $\mathbb{E}\hat{f}_{C^{g*}}(x; n) = f_{C^{g*}}(x)$ for each $x \in \mathbb{R}$,
2) $\mathbb{P}\Big(\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |\hat{f}_{C^{g*}}(x; n) - f_{C^{g*}}(x)|\Big) = 1$,
3) $\mathbb{E}h(C^{g*}) = \mathbb{E}\left[\int_{\mathbb{R}} h(x)\hat{f}_{C^{g*}}(x; n)dx\right]$ for each measurable $h : \mathbb{R} \mapsto [0, \infty)$.

6.2 Empirical distributions of C^{e*} and C^{g*}

As mentioned in Section 4, it is natural to conjecture that the typical shortest–path lengths $C^{e*} = C^{e*}(\gamma, \lambda_\ell, \lambda'_\ell)$ are stochastically decreasing in λ_ℓ. In Figure 7, we have plotted densities and distribution functions of the estimated typical shortest–path length C^{e*} for the case of e-closeness, where G is the edge set of a PVT and the estimator $\hat{f}_{C^{e*}}(x; n)$ for the density of C^{e*} introduced in [18] has been used. The four plots correspond to $\kappa = 10, 20, 50, 100$, respectively (where we fixed $\gamma = 1$ and $n = 2000$ iterations). It is clearly

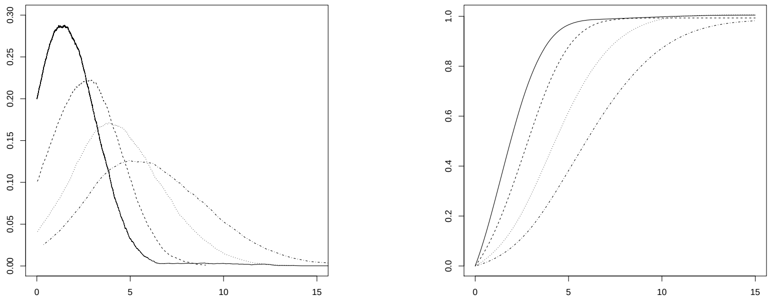

Fig. 7 Densities (left) and distribution functions (right) of C^{e*} for PVT where $\kappa = 10$ (solid), $\kappa = 20$ (dashed), $\kappa = 50$ (dotted), $\kappa = 100$ (dotdashed).

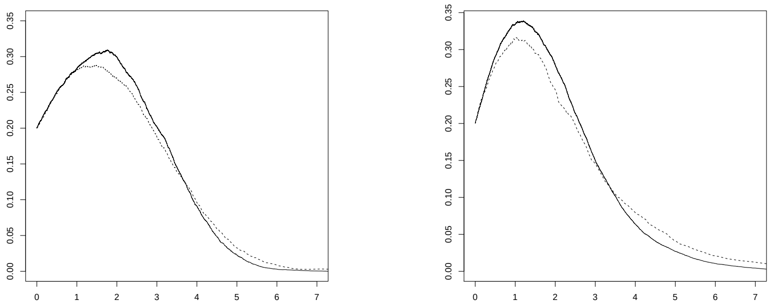

Fig. 8 Densities of C^{e*} (dashed) and C^{g*} (solid) for PVT (left) and PLT (right) where $\kappa = 10$

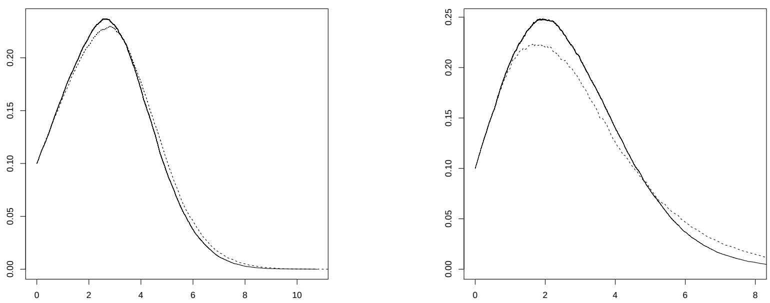

Fig. 9 Densities of C^{e*} (dashed) and C^{g*} (solid) for PVT (left) and PLT (right) where $\kappa = 20$

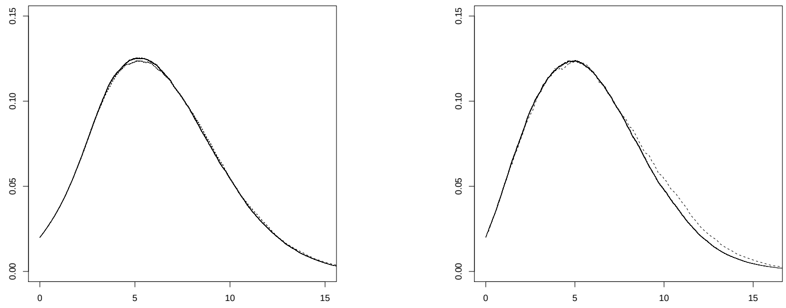

Fig. 10 Densities of C^{e*} (dashed) and C^{g*} (solid) for PVT (left) and PLT (right) where $\kappa = 100$

κ	$\mathbb{E}C^{e*}$	$\mathbb{E}C^{g*}$
10	2.117	1.966
20	2.930	2.788
100	6.235	6.169

κ	$\mathbb{E}C^{e*}$	$\mathbb{E}C^{g*}$
10	2.155	1.866
20	2.914	2.710
100	6.208	5.855

Table 1 Expectations of C^{e*} and C^{g*} for PVT (left) and PLT (right)

visible that the distribution functions on the right–hand side of Figure 7 do not intersect and that each pair of density functions on the left–hand side of Figure 7 intersect only once. These two observations provide strong indications for the stochastic monotonicity of the respective typical shortest–path lengths C^{e*}. Similar results have been obtained for other classes of stationary tessellations with convex cells, see also [17,18] for PLT and PDT. But we were not able to provide a proof of this monotonicity property. However, recall that in contrast to this situation for e-closeness, stochastic monotonicity of $C^{g*} = C^{g*}(\gamma, \lambda_\ell, \lambda'_\ell)$ can formally be shown, see Proposition 4.4.

Furthermore, recall that in Proposition 4.2 we showed that C^{e*} is always stochastically larger than C^{g*}. We evaluated the simulation–based density estimators $\hat{f}_{C^{g*}}(x; n)$ and $\hat{f}_{C^{e*}}(x; n)$ mentioned in Section 6.1 for $n = 2000$ simulations, in order to find out how much larger C^{e*} is than C^{g*} for given specifications of the parameter vector $(\gamma, \lambda_\ell, \lambda'_\ell)$. From Figures 8 and 9 we see that the empirical densities $\hat{f}_{C^{g*}}(x; n)$ and $\hat{f}_{C^{e*}}(x; n)$ are quite different from each other for $\kappa = 10$ and $\kappa = 20$, whereas for $\kappa = 100$ this is not the case (at least not for PVT), see Figure 10. Note that the latter phenomenon is in accordance with the result of Theorem 5.2, which states that the distributions of C^{e*} and C^{g*} converge to the same limit as $\kappa \to \infty$.

Besides this, we compared the expectations of C^{e*} and C^{g*} for PVT and PLT, see Table 1. In both cases, we can observe that the estimated values of $\mathbb{E}C^{e*}$ are larger than those obtained for $\mathbb{E}C^{g*}$, where the differences are much larger for PLT than for PVT. Furthermore, it is interesting to see that if we pass from the PVT to the PLT case, the estimated expectation of C^{g*} always decreases while for C^{e*} this remains no longer true.

7 Conclusions

We have developed two different approaches to the problem of connecting high–level and low–level components in hierarchical network models, where two different meanings of 'closeness' are considered: either with respect to the Euclidean distance (e-closeness), or in a graph–theoretic sense, i.e., along the edges of the network (g-closeness). Furthermore, we extended the class of stationary graph models describing the infrastructure of the network, admitting e.g. tessellations with non–convex cells and graphs with dead ends. We proved stochastic comparability and monotonicity properties of typical shortest–path lengths between the locations of high–level and low–level network components and we determined their asymptotic limit distributions for unboundedly sparse and dense networks, respectively.

However, some open problems remain to be solved. This will be the subject of future research. For example, it would be interesting to prove (or disprove) that the typical shortest–path length $C^{e*}(\gamma, \lambda_\ell, \lambda'_\ell)$ stochastically decreases in λ_ℓ for a large class of stationary random geometric graphs G. Another interesting problem would be to find out whether (resp. under which conditions) the asymptotic behavior considered in Theorem 5.2 remains true if G is the edge set of an aggregate tessellation induced by non–Poissonian point processes, e.g., by Poisson cluster processes or Cox processes. Furthermore, the same problem could be investigated in the case that $G = G(\beta, X)$ is a β-skeleton induced by a stationary (non–Poissonian) point process X. Finally, for practical applications of Theorem 5.2 to real network data, it would be useful to determine the constant ξ appearing in this theorem nmuerically, e.g., for aggregate tessellations and for other classes of stationary and connected random geometric graphs.

Acknowlegdements The authors would like to thank the two anonymous referees for their careful reading of an earlier version of the manuscript. Their suggestions helped to improve the presentation of the material. This work has been supported by Orange Labs through Research grant No. 46146063-9241. Christian Hirsch has been supported by a research grant from DFG Research Training Group 1100 at Ulm University.

References

1. D. Aldous and J. Shun. Connected spatial networks over random points and a route-length statistic. *Statistical Science* **25** (2010), 275-288.
2. F. Baccelli, G. Gloaguen, and S. Zuyev. Superposition of planar Voronoi tessellations. *Stochastic Models* **16** (2000), 69–98.

3. P. Bose, L. Devroye, W. Evans, and D. Kirkpatrick. On the spanning ratio of gabriel graphs and β-skeletons. In: Proceedings of the 5th Latin American Symposium on Theoretical Informatics (LATIN'02), Lecture Notes in Computer Science **2286**, Springer, Berlin, 2002, 479–493.
4. P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the crofton cell in the plane. *Advances in Applied Probability* **34** (2002), 702–717.
5. D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Vol. I/II, Springer, New York, 2005/08.
6. R. Durrett *Probability: Theory and Examples.* 2nd edition, Duxbury Press, Belmont, 1996.
7. S.G. Foss and S.A. Zuyev. On a Voronoi aggregative process related to a bivariate Poisson process. *Advances in Applied Probability* **28** (1996), 965–981.
8. C. Gloaguen, F. Voss, and V. Schmidt. Parametric distributions of connection lengths for the efficient analysis of fixed access network. *Annals of Telecommunications* **66** (2011), 103–118.
9. J. Illian, A. Penttinen, H. Stoyan, and D. Stoyan. *Statistical Analysis and Modelling of Spatial Point Patterns.* J. Wiley & Sons, Chichester, 2008.
10. D.G. Kirkpatrick and J.D. Radke. A framework for computational morphology. In: G.T. Toussaint (ed.) *Computational Geometry.* North Holland, Amsterdam 1985, 217–248.
11. I.S. Molchanov. *Theory of Random Sets.* Springer, London, 2005.
12. J. Neveu. Processus ponctuels. In *École d'Été de Probabilités de Saint-Flour VI. Lecture Notes in Mathematics* **598** (1977), Springer, 249–445.
13. M. Penrose. *Random Geometric Graphs.* Oxford University Press, Oxford, 2003.
14. R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer, Berlin, 2008.
15. D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic Geometry and its Applications.* J. Wiley & Sons, Chichester, 1995.
16. K. Tchoumatchenko and S. Zuyev. Aggregate and fractal tessellations. *Probability Theory and Related Fields* **121** (2001), 198–218.
17. F. Voss, C. Gloaguen, and V. Schmidt. Scaling limits for shortest path lengths along the edges of stationary tessellations. *Advances in Applied Probability* **42** (2010), 936-952.
18. F. Voss, C. Gloaguen, F. Fleischer, and V. Schmidt. Densities of shortest path lengths in spatial stochastic networks. *Stochastic Models* **27** (2011), 141–167.