Applied Probability Trust (22nd September 2004)

LIMIT THEOREMS FOR STATIONARY TESSELLATIONS WITH RANDOM INNER CELL STRUCTURES

LOTHAR HEINRICH,* *University of Augsburg*
HENDRIK SCHMIDT ** AND
VOLKER SCHMIDT,** *University of Ulm*

Abstract

Stationary and ergodic tessellations $X = \{\Xi_n\}_{n\geq 1}$ in $\mathbb{R}^d$ are considered, where X is observed in a bounded and convex sampling window $W_\varrho \subset \mathbb{R}^d$. It is assumed that the cells Ξ_n of X possess random inner structures, examples of which include point patterns, fibre systems, and tessellations. These inner cell structures are generated both independently of each other and independently of the tessellation X by generic stationary random sets which are connected with a stationary random vector measure J_0 acting on $\mathbb{R}^d$. In particular, the asymptotic behavior of a multivariate random functional is studied, which is determined both by X and the individual cell structures contained in W_ϱ, as $W_\varrho \uparrow \mathbb{R}^d$. It turns out that by this functional an unbiased estimator for the intensity vector associated with J_0 is provided. Furthermore, under natural restrictions, strong laws of large numbers and a multivariate central limit theorem of the normalized functional are proven. Finally, some numerical examples and applications are discussed in detail, for which the inner structures of the cells of X are induced by iterated Poisson-type tessellations.

Keywords: Stationary random tessellations; inner random cell structure; cumulative random functionals; ergodic theorem; central limit theorem; Poisson nesting; model choice; telecommunication networks

AMS 2000 Subject Classification: Primary 60D05
Secondary 60F05; 90B15

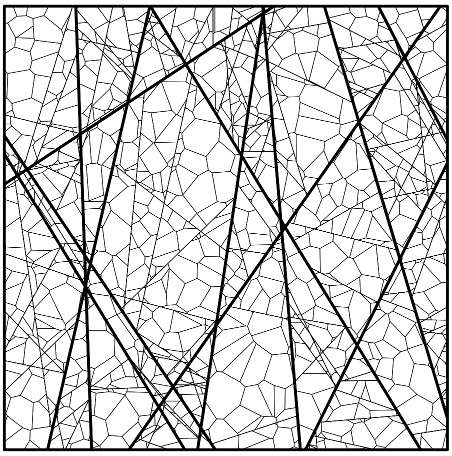
1. Introduction

Let $X = \{\Xi_n\}_{n\geq 1}$ be a stationary and ergodic tessellation in $\mathbb{R}^d$. For each $n \geq 1$, consider a certain (d–dimensional) random vector $\alpha(\Xi_n)$, which is called an associated point of the cell Ξ_n of X. It is well known that the tessellation X can be regarded as a stationary and ergodic marked point process $\sum_{n\geq 1} \delta_{[\alpha(\Xi_n),\Xi_n^0]}$ in $\mathbb{R}^d$, where the shifted cells $\Xi_n^0 = \Xi_n - \alpha(\Xi_n)$ contain the origin $o \in \mathbb{R}^d$; see, e.g., Section 6.1 of [22]. Furthermore, for each $n \geq 1$, consider a vector $J_n = \left(J_n^{(1)}, \ldots, J_n^{(m)}\right)^\top$ of $m \geq 1$ stationary random measures in $\mathbb{R}^d$. Assume that the sequence $(J_n)_{n\geq 1}$ is independent of X and that it consists of i.i.d. copies of some generic random vector measure $J_0 = (J_0^{(1)}, \ldots, J_0^{(m)})^\top$.

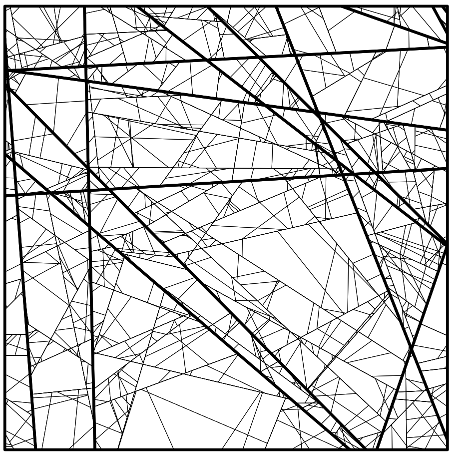
For each $n \geq 1$, the random measures $J_n^{(1)}, \ldots, J_n^{(m)}$ describe the inner structure of the n-th cell Ξ_n of the tessellation X. In particular, in the planar case $d = 2$, examples

* Postal address: Institute of Mathematics, University of Augsburg, D-86135 Augsburg, Germany
** Postal address: Department of Stochastics, University of Ulm, D-89069 Ulm, Germany

of such random measures are the number of vertices and the number or the total length of the edges, which are generated by some (component) tessellation X_n within the cell Ξ_n of the (initial) tessellation X. Figure 1(a) shows the case where X_n is a so–called PLT/PVT–nesting, which means that Poisson line tessellations (PLT) iterated by Poisson–Voronoi tessellations (PVT) are inscribed into the cells of X. Similarly in Figure 1(b), PLT/PLT–nestings inscribed into the cells of X are shown. However, the random measures $J_n^{(1)}, \ldots, J_n^{(m)}$ need not necessarily be induced by tessellations. Another type of examples is shown in Figure 2, where the inner structure of Ξ_n is determined by point processes.



(a) *PLT/PVT–nesting*



(b) *PLT/PLT–nesting*

FIGURE 1: *Inner structure of the cells of X induced by iterated tessellations*

Suppose that only a single realization of the tessellation X as well as of the vector measures $J_1, J_2, \ldots$ is available. This realization is restricted to some (presumably large) sampling window W_ϱ and the support of J_n is observable only in $\Xi_n \cap W_\varrho$ for $n \geq 1$. The region W_ϱ is assumed to have the form $W_\varrho = \varrho W$ with scaling factor $\varrho > 0$ (which increases unboundedly) and with a convex body $W \subset \mathbb{R}^d$ containing the closed ball $b(o, r)$ centered at the origin with fixed radius $r > 0$. The main subject investigated in the present paper is the vector of cumulative functionals $Z_\varrho = (Z_\varrho^{(1)}, \ldots, Z_\varrho^{(m)})^\top$, where the components $Z_\varrho^{(i)}$ of Z_ϱ are given by

$$Z_\varrho^{(i)} = \sum_{n \geq 1} J_n^{(i)}(\Xi_n \cap W_\varrho)\,, \tag{1.1}$$

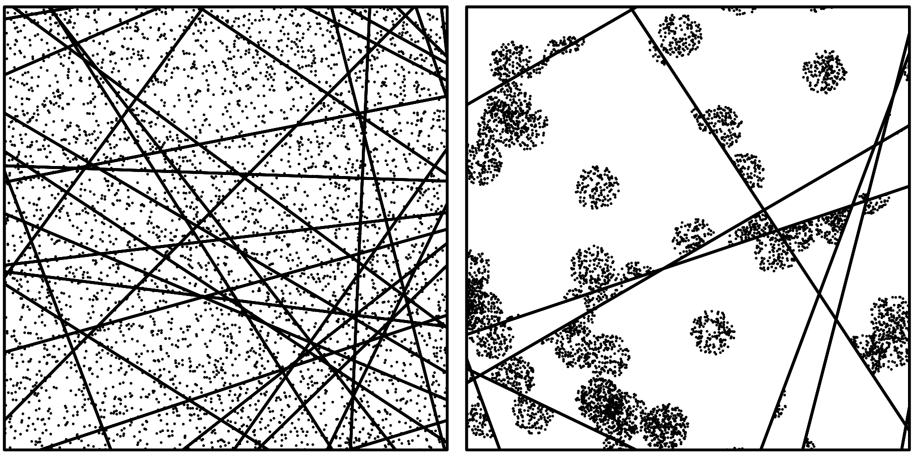
for each $i = 1, \ldots, m$. In the first step, the expectation vector $\mathbb{E} Z_\varrho$ and the covariance matrix $\mathsf{Cov}(Z_\varrho)$ of Z_ϱ as well as the asymptotic covariance matrix $K = \lim_{\varrho \to \infty} \mathsf{Cov}(\widetilde{Z}_\varrho)$ of the vector of normalized functionals

$$\widetilde{Z}_\varrho = \left(\frac{Z_\varrho^{(1)} - \lambda^{(1)}\,|W_\varrho|}{\sqrt{|W_\varrho|}}\,, \ldots, \,\frac{Z_\varrho^{(m)} - \lambda^{(m)}\,|W_\varrho|}{\sqrt{|W_\varrho|}} \right)^\top \tag{1.2}$$

are determined. Here, $|B|$ denotes the d-dimensional Lebesgue measure of the bounded Borel set $B \subset \mathbb{R}^d$ and $\lambda^{(i)} = \mathbb{E} J_0^{(i)}([0,1)^d)$ is the intensity of the stationary random measure $J_0^{(i)}$; see Theorem 3.1. In the next step, under some mild integrability conditions, Theorem 4.2 yields the following strong law of large numbers

$$\frac{1}{|W_\varrho|} \sum_{n \geq 1} J_n^{(i)}(\Xi_n \cap W_\varrho) \underset{\varrho \to \infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda^{(i)} \quad \text{for} \quad i = 1, \ldots, m\,,$$

showing that the vector $Z_\varrho / |W_\varrho|$ is a strongly consistent (and unbiased) estimator for the intensity vector $(\lambda^{(1)}, \ldots, \lambda^{(m)})^\top$ of the stationary vector measure J_0. The proof of Theorem 4.2 relies on the ergodicity of the tessellation $X = \{\Xi_n\}_{n \geq 1}$ and the conditional independence of the random vectors $J_1(\Xi_1 \cap W_\varrho), J_2(\Xi_2 \cap W_\varrho), \ldots$ given the tessellation X. For this purpose some estimates are needed which show that the contribution of those cells of X hitting the boundary ∂W_ϱ is asymptotically negligible as $\varrho \to \infty$; see Lemma 4.1.



(a) *Poisson point process* (b) *Cluster point process*

FIGURE 2: *Inner structure of the cells of X induced by point processes*

In the third step, the following multivariate central limit theorem

$$\widetilde{Z}_\varrho \underset{\varrho \to \infty}{\Longrightarrow} \mathrm{N}(o, K)\,, \qquad \text{i.e.,} \qquad \lim_{\varrho \to \infty} \sup_{x \in \mathbb{R}^m} \left| \mathbb{P}\big(\widetilde{Z}_\varrho \leq x\big) - \Phi_K(x) \right| = 0$$

is derived, where „$\Longrightarrow$" means convergence in distribution and Φ_K denotes the distribution function of the (m-dimensional) Gaussian vector $\mathrm{N}(o, K)$ with zero mean components and covariance matrix K; see Theorem 5.1.

Our results can be applied to stochastic modelling and statistical analysis of complex network structures. In [9] the so–called stochastic subscriber line model (SSLM) is described, which is an example of a stochastic–geometric model of telecommunication networks. Figure 3 shows a realization of the SSLM, where the urban infrastructure along which the cable trench system is built is represented by a random tessellation. Within each cell, subscribers are located according to some point process and line segments represent dead end streets. Along the streets 1–level and 2–level stations are displayed, where each subscriber is connected to its closest 2–level station via a 1–level station.

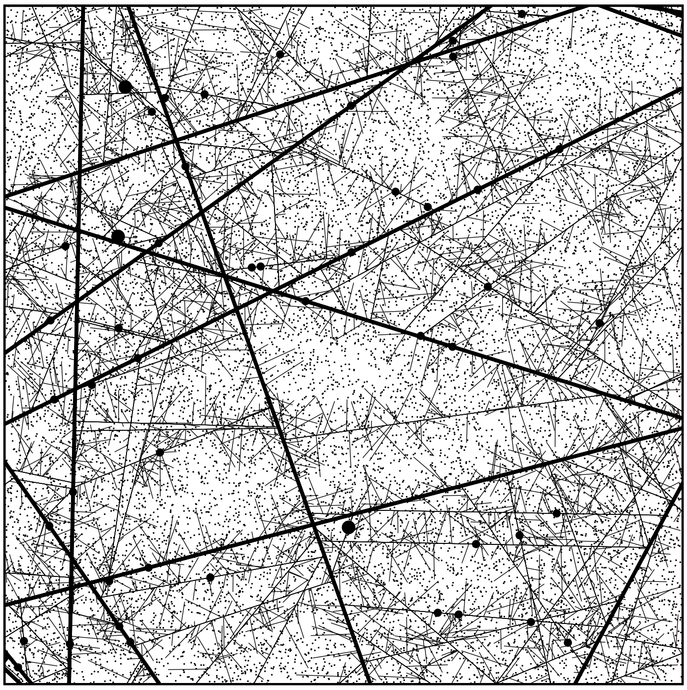


FIGURE 3: *Realization of the stochastic subscriber line model.*

In the context of the SSLM, the results of the present paper provide a theoretical basis for statistical analysis of the morphological structure of spatial telecommunication data and help fit appropriate tessellation models.

Notice that there exists a number of papers investigating problems closely related to the topics of our work. For example, [11] derives a central limit theorem for a class of random measures associated with germ–grain models, while [1] and [19] investigate central limit theorems for Poisson–Voronoi and Poisson line tessellations in $\mathbb{R}^2$, respectively. In [10], normal approximations are given for some mean–value estimates of absolutely regular tessellations. Asymptotic properties of estimators for the volume fraction and other specific intrinsic volumes of stationary random sets are examined in [3], [4], [16], and [21], for example. Simulation studies on the typical cell of stationary tessellations can be found, e.g., in [13].

The present paper is organized as follows. In Section 2, a short introduction on basic notions and notations of stochastic geometry in general is given. Section 3 is devoted to first and second order moments for functionals of stationary random measures associated with the cells of random tessellations. The strong law of large numbers and the multivariate central limit theorem mentioned above are derived in Sections 4 and 5, respectively. Some numerical examples are discussed in Section 6, where functionals are considered which describe several intracellular structures of the cells of tessellations in $\mathbb{R}^2$. Finally, in Section 7 an outlook on further research perspectives is given.

2. Basic notions and notation

In this section, the basic notation used in the present paper is introduced and a brief account of some relevant notions of stochastic geometry is given. For a detailed discussion of the subject, it is referred to the literature, for example [22] and [23].

The abbreviations int $B, \partial B$, and B^c are used to denote the interior, the boundary, and the complement of a set $B \subset \mathbb{R}^d$, respectively. For arbitrary sets $B, B' \in \mathbb{R}^d$, we will consider the operations of translation $B + x = \{y + x : y \in B\}$ for $x \in \mathbb{R}^d$, reflection $\check{B} = -B = \{-x : x \in B\}$, scaling $B_\varrho = \{\varrho x : x \in B\}$ for any constant $\varrho \in [0, \infty)$, and Minkowski–addition $B \oplus B' = \{x + x' : x \in B, x' \in B'\}$. Furthermore, let $b(x, r) = \{y \in \mathbb{R}^d : \|x - y\| \le r\}$ denote the ball of radius $r \ge 0$ centered at $x \in \mathbb{R}^d$, where $\|x - y\|$ is the length of the vector $x - y$.

By $\mathcal{F}$, $\mathcal{K}$, and $\mathcal{C}$, the families of all closed sets, compact sets, and convex bodies (compact and convex sets) in $\mathbb{R}^d$ are denoted, respectively. For any $C \in \mathcal{C}$, the volume $|C \oplus b(o, r)|$ of the so–called parallel set $C \oplus b(o, r)$ is given by Steiner's formula

$$|C \oplus b(o,r)| = \sum_{k=0}^{d} \binom{d}{k} M_k(C)\, r^k\,, \qquad r \ge 0\,, \tag{2.1}$$

where $M_k(C)$ denotes the k-th Minkowski–functional of the convex body C; $0 \le k \le d$. Note that in particular $M_0(C) = |C|$ and $M_d(C) = 1 - \delta_\emptyset(C)$. Clearly, for each $\varrho > 0$ and for any convex body $C \in \mathcal{C}$, the volume $|C_\varrho|$ equals $\varrho^d|C|$, and the diameter $D(C_\varrho)$ of the scaled set C_ϱ is given by $\varrho D(C)$, where $D(C) = \sup\{\|x - y\| : x, y \in C\}$. Furthermore, for any $C \in \mathcal{C}$, the following isodiametric inequality holds (see e.g. [12])

$$M_k(C) \le \omega_d \left(\frac{D(C)}{2}\right)^{d-k}, \qquad 0 \le k \le d\,, \tag{2.2}$$

where ω_d denotes the volume of the unit ball $b(o, 1)$. Recall that a random closed set Ξ in $\mathbb{R}^d$ is a measurable mapping $\Xi : \Omega \to \mathcal{F}$ from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$, where $\mathcal{B}(\mathcal{F})$ denotes the smallest σ–algebra of subsets of $\mathcal{F}$ that contains all sets $\{F \in \mathcal{F}, F \cap K \ne \emptyset\}$ for any $K \in \mathcal{K}$. Particularly, the random closed set Ξ is called a random compact set or a random convex body if $\mathbb{P}(\Xi \in \mathcal{K}) = 1$ or $\mathbb{P}(\Xi \in \mathcal{C}) = 1$, respectively.

A tessellation in $\mathbb{R}^d$ is a countable family $\tau = \{C_n\}_{n\ge1}$ of convex bodies $C_n \in \mathcal{C}$ such that int $C_n \ne \emptyset$ for all n, int $C_n \cap$ int $C_m = \emptyset$ for all $n \ne m$, $\bigcup_{n\ge1} C_n = \mathbb{R}^d$, and $\sum_{n\ge1} \mathbb{I}_{\{C_n \cap K \ne \emptyset\}} < \infty$ for any $K \in \mathcal{K}$. Notice that the sets C_n, called the cells of τ, are polytopes in $\mathbb{R}^d$. The family of all tessellations in $\mathbb{R}^d$ is denoted by $\mathcal{T}$. A random tessellation $\{\Xi_n\}_{n\ge1}$ in $\mathbb{R}^d$ is a sequence of random convex bodies Ξ_n such that $\mathbb{P}(\{\Xi_n\}_{n\ge1} \in \mathcal{T}) = 1$. Notice that a random tessellation $\{\Xi_n\}_{n\ge1}$ can also be considered as a marked point process $\sum_{n\ge1} \delta_{[\alpha(\Xi_n), \Xi_n^0]}$, where $\alpha : \mathcal{C}' \to \mathbb{R}^d$, $\mathcal{C}' = \mathcal{C}\setminus\{\emptyset\}$, is a measurable mapping such that $\alpha(C) \in C$ and $\alpha(C + x) = \alpha(C) + x$ for any $C \in \mathcal{C}'$ and $x \in \mathbb{R}^d$, and where $\Xi_n^0 = \Xi_n - \alpha(\Xi_n)$ is the centered cell corresponding to Ξ_n which contains the origin. The point $\alpha(C) \in \mathbb{R}^d$ is called the associated point of C and can be chosen, for example, to be the lexicographically smallest point of C.

Suppose that the marked point process $\sum_{n\geq 1}\delta_{[\alpha(\Xi_n),\Xi_n^0]}$ is stationary with positive and finite intensity $\lambda = \mathbb{E}\#\{n : \alpha(\Xi_n)\in[0,1)^d\}$. By $\mathcal{P}^0$ we denote the set of all convex polytopes with their associated point at the origin. Then, the Palm mark distribution P^0 of X is given by

$$P^0(B) = \lambda^{-1}\,\mathbb{E}\#\{n : \alpha(\Xi_n)\in[0,1)^d,\ \Xi_n^0\in B\}\,, \qquad B\in\mathcal{B}(\mathcal{F})\cap\mathcal{P}^0\,. \tag{2.3}$$

Notice that a random polytope $\Xi^* : \Omega\to\mathcal{P}^0$, whose distribution coincides with P^0, is called the typical cell of X. Furthermore, it holds

$$\lambda^{-1} = \int_{\mathcal{P}^0}|C|\,P^0(dC)\,, \tag{2.4}$$

i.e., the mean volume $\mathbb{E}|\Xi^*| = \int_{\mathcal{P}^0}|C|\,P^0(dC)$ of the typical cell Ξ^* is equal to λ^{-1}.

A (deterministic) iterated tessellation $\tau = \{C_{n\nu}\cap C_n : \text{int } C_{n\nu}\cap\text{int } C_n\neq\emptyset\}$ in $\mathbb{R}^d$ consists of an initial tessellation $\tau = \{C_n\}_{n\geq 1}$ in $\mathbb{R}^d$ and a sequence $(\tau_n)_{n\geq 1}$ of component tessellations $\tau_n = \{C_{n\nu}\}_{\nu\geq 1}$. Hence, in order to define the notion of a random iterated tessellation, we can proceed as follows; see [14]. Let Ξ be a random convex body in $\mathbb{R}^d$, where int $\Xi\neq\emptyset$, and let $X = \{\Xi_n\}_{n\geq 1}$ be a random tessellation in $\mathbb{R}^d$. Then, the mapping $Y(\cdot\mid\Xi) : \Omega\to N(\mathcal{F}')$ defined by $Y(B\mid\Xi) = \sum_{n\geq 1}\delta_{\Xi_n\cap\Xi}(B)\,\mathbb{I}_{\{\text{int }\Xi_n\cap\text{int }\Xi\neq\emptyset\}}$ for $B\in\mathcal{B}(\mathcal{F}')$ is a point process in $\mathcal{C}'$, where $\mathcal{F}' = \mathcal{F}\setminus\{\emptyset\}$. The space of all non–negative and integer–valued measures on $\mathcal{B}(\mathcal{F}')$ is denoted by $N(\mathcal{F}')$, where each $\eta\in N(\mathcal{F}')$ can be represented by a finite or countable sum of Dirac measures δ_F of sets $F\in\mathcal{F}'$, i.e., $\eta(B) = \sum_{n\geq 1}\eta(\{F_n\})\delta_{F_n}(B)$ for any $B\in\mathcal{B}(\mathcal{F}')$, and that $\eta(\{F\in\mathcal{F} : F\cap K\neq\emptyset\}) < \infty$ for any $K\in\mathcal{K}$. Notice that $Y(\cdot\mid\Xi)$ can be seen as one possible way to describe a random tessellation in Ξ.

Furthermore, if $X = \{\Xi_n\}_{n\geq 1}$ is an arbitrary random tessellation in $\mathbb{R}^d$ and if $\{X_n\}_{n\geq 1}$ is an independent sequence of independent and identically distributed random tessellations $X_n = \{\Xi_{n\nu}\}_{\nu\geq 1}$ in $\mathbb{R}^d$, then the mapping $Y : \Omega\to N(\mathcal{F}')$ defined by $Y(B) = \sum_n Y_n(B\mid\Xi_n)$ and $Y_n(B\mid\Xi_n) = \sum_{\nu\geq 1}\delta_{\Xi_{n\nu}\cap\Xi_n}(B)\,\mathbb{I}_{\{\text{int }\Xi_{n\nu}\cap\text{int }\Xi_n\neq\emptyset\}}$ for $B\in\mathcal{B}(\mathcal{F}')$ is called the point–process representation of an iterated random tessellation (or X/X_n–nesting) in $\mathbb{R}^d$ with initial tessellation X and component tessellations $X_1, X_2, \ldots$. Clearly, the point process Y is stationary and isotropic, respectively, provided that both the initial tessellation X and the component tessellations $X_1, X_2, \ldots$ possess these properties. Moreover, Y is ergodic if X is ergodic.

3. Expectation vector and covariance matrix

Let $X = \{\Xi_n\}_{n\geq 1}$ be an arbitrary stationary and ergodic tessellation in $\mathbb{R}^d$. Recall that the tessellation X can be equivalently described as a stationary and ergodic marked point process $\sum_{n\geq 1}\delta_{[\alpha(\Xi_n),\Xi_n^0]}$, where $\Xi_n^0 = \Xi_n - \alpha(\Xi_n)$ and $\alpha(\Xi_n)$ denotes the associated point of Ξ_n. The intensity $\lambda = \mathbb{E}\#\{n : \alpha(\Xi_n)\in[0,1)^d\}$ is assumed to be positive and finite. For each individual cell Ξ_n of X, we consider an m–dimensional vector $J_n = \big(J_n^{(1)},\ldots,J_n^{(m)}\big)^\top$ of stationary random measures in $\mathbb{R}^d$, which describe the inner structure of Ξ_n. We assume that the sequence $(J_n)_{n\geq 1}$ is independent of X and consists of i.i.d. copies of a generic stationary random vector measure $J_0 = (J_0^{(1)},\ldots,J_0^{(m)})^\top$.

Throughout this paper, we assume that only a single realization of the tessellation X and of the random vectors $J_n(\Xi_n\cap W_\varrho)$, $n\geq 1$, can be observed in an (unboundedly increasing) sampling window $W_\varrho = \varrho W \uparrow \mathbb{R}^d$ (as $\varrho \uparrow \infty$), where the convex body W satisfies the inclusion $b(o,r) \subseteq W \subseteq b(o,R)$ for some fixed $0 < r < R < \infty$. To begin with, we determine the expectation vector $\mathbb{E}Z_\varrho$ of the random vector $Z_\varrho = (Z_\varrho^{(1)}, \dots, Z_\varrho^{(m)})^\top$ defined by (1.1). Furthermore, we derive conditions under which the covariance matrix $\mathsf{Cov}(Z_\varrho)$ and the limit $\lim_{\varrho\to\infty} \mathsf{Cov}(\widetilde{Z}_\varrho)$ exist, where $\widetilde{Z}_\varrho$ is the normalized vector of functionals introduced in (1.2).

Theorem 3.1. *If* $\lambda^{(i)} = \mathbb{E}J_0^{(i)}([0,1)^d) < \infty$ *for each* $i = 1, \dots, m$, *then*

$$\mathbb{E}Z_\varrho = |W_\varrho|(\lambda^{(1)}, \dots, \lambda^{(m)})^\top . \tag{3.1}$$

Under the additional assumption that

$$\int_{\mathcal{P}^0} \mathbb{E}\big(J_0^{(i)}(C)\big)^2 \, P^0(dC) < \infty , \qquad i = 1, \dots, m , \tag{3.2}$$

the covariance matrix $\mathsf{Cov}(Z_\varrho) = \Big(\mathrm{Cov}(Z_\varrho^{(i)}, Z_\varrho^{(j)})\Big)_{i,j=1}^m$ *exists with entries taking the form*

$$\mathrm{Cov}(Z_\varrho^{(i)}, Z_\varrho^{(j)}) = \lambda \int_{\mathcal{P}^0} \int_{\mathbb{R}^d} \mathrm{Cov}\Big(J_0^{(i)}(C \cap (W_\varrho - x)), J_0^{(j)}(C \cap (W_\varrho - x))\Big) \, dx \, P^0(dC) . \tag{3.3}$$

Moreover, the asymptotic covariance matrix $K = \lim_{\varrho\to\infty} \mathsf{Cov}(\widetilde{Z}_\varrho) = \Big(\sigma_{ij}^2\Big)_{i,j=1}^m$ *exists with entries*

$$\sigma_{ij}^2 = \lambda \int_{\mathcal{P}^0} \mathrm{Cov}\Big(J_0^{(i)}(C), J_0^{(j)}(C)\Big) P^0(dC) , \qquad i,j = 1, \dots, m . \tag{3.4}$$

Proof. In view of the independence of X and the sequence $(J_n)_{n\geq 1}$ we may write for any $i = 1, \dots, m$ that

$$\mathbb{E}Z_\varrho^{(i)} = \mathbb{E}\Big(\sum_{n\geq 1} J_n^{(i)}(\Xi_n \cap W_\varrho)\Big) = \mathbb{E}\Big(\sum_{n\geq 1} \mathbb{E}_X\Big(J_0^{(i)}(\Xi_n \cap W_\varrho)\Big)\Big) ,$$

where $\mathbb{E}_X$ denotes the conditional expectation given the tessellation X. By stationarity of $J_0^{(i)}$, the expectation $\mathbb{E}_X\big(J_0^{(i)}(\Xi_n \cap W_\varrho)\big)$ equals $\lambda^{(i)}\,|\Xi_n \cap W_\varrho|$. Since the interior of the cells Ξ_n, $n \geq 1$ fills the space $\mathbb{R}^d$ up to a set of Lebesgue measure zero, we have

$$\sum_{n\geq 1} |\Xi_n \cap W_\varrho| \;=\; |W_\varrho| , \tag{3.5}$$

which proves (3.1). To derive (3.3) we first carry out all rearrangements without regarding the existence of the integrals and expectations involved and after that we will check their correctness. Using the independence of X and the sequence $(J_n)_{n\geq 1}$ once more and combining this with (3.1), we get

$$\mathrm{Cov}\Big(Z_\varrho^{(i)}, Z_\varrho^{(j)}\Big) = \mathbb{E}\Big(\sum_{n\geq 1} \mathbb{E}_X\Big(J_0^{(i)}(\Xi_n \cap W_\varrho) J_0^{(j)}(\Xi_n \cap W_\varrho)\Big)\Big) \quad +$$

$$\mathbb{E}\Big(\sum_{\substack{n,\ell\geq 1\\ n\neq\ell}} \mathbb{E}_X\Big(J_n^{(i)}(\Xi_n\cap W_\varrho)J_\ell^{(j)}(\Xi_\ell\cap W_\varrho)\Big)\Big) \;-\; \lambda^{(i)}\,\lambda^{(j)}\,|W_\varrho|^2\,.$$

The difference of the two expressions in the latter line is just equal to

$$-\,\mathbb{E}\Big(\sum_{n\geq 1}\mathbb{E}_X\Big(J_0^{(i)}(\Xi_n\cap W_\varrho)\Big)\mathbb{E}_X\Big(J_0^{(j)}(\Xi_n\cap W_\varrho)\Big)\Big)\,,$$

since, for $n\neq\ell$, we have

$$\mathbb{E}_X\Big(J_n^{(i)}(\Xi_n\cap W_\varrho)J_\ell^{(j)}(\Xi_\ell\cap W_\varrho)\Big) = \mathbb{E}_X J_n^{(i)}(\Xi_n\cap W_\varrho)\mathbb{E}_X J_\ell^{(j)}(\Xi_\ell\cap W_\varrho)$$

by the assumed independence of $J_n^{(i)}$ and $J_\ell^{(i)}$. Thus, denoting by Cov_X the conditional covariance given the tessellation $X=\{\Xi_n\}_{n\geq 1}$, we find that

$$\mathrm{Cov}\Big(Z_\varrho^{(i)},Z_\varrho^{(j)}\Big) = \mathbb{E}\Big(\sum_{n\geq 1}\mathrm{Cov}_X\Big(J_0^{(i)}(\Xi_n\cap W_\varrho),J_0^{(j)}(\Xi_n\cap W_\varrho)\Big)\Big)\,. \tag{3.6}$$

Finally, writing $\Xi_n=\Xi_n^0+\alpha(\Xi_n)$ and applying Campbell's theorem to the stationary marked point process $\sum_{n\geq 1}\delta_{[\alpha(\Xi_n),\Xi_n^0]}$, we get

$$\begin{aligned}\mathrm{Cov}\Big(Z_\varrho^{(i)},Z_\varrho^{(j)}\Big) &= \lambda\int_{\mathbf{R}^d}\int_{\mathcal{P}^0}\mathrm{Cov}\Big(J_0^{(i)}((C+x)\cap W_\varrho),J_0^{(j)}((C+x)\cap W_\varrho)\Big)P^0(dC)\,dx\\ &= \lambda\int_{\mathcal{P}^0}\int_{\mathbf{R}^d}\mathrm{Cov}\Big(J_0^{(i)}(C\cap(W_\varrho-x)),J_0^{(j)}(C\cap(W_\varrho-x))\Big)\,dx\,P^0(dC)\,,\end{aligned}$$

where in the latter equality we used Fubini's theorem and the invariance of the covariance $\mathrm{Cov}\Big(J_0^{(i)}(A),J_0^{(j)}(B)\Big)$ under diagonal shifts, i.e. $\mathrm{Cov}\Big(J_0^{(i)}(A+x),J_0^{(j)}(B+x)\Big) = \mathrm{Cov}\Big(J_0^{(i)}(A),J_0^{(j)}(B)\Big)$ for any $A,B\in\mathcal{C}$ and for any $x\in\mathbb{R}^d$. To complete the proof of (3.3), we justify the steps and changes of integration above by showing that our integrability condition (3.2) ensures the existence of the second moment

$$\mathbb{E}\sum_{n\geq 1}\Big(J_n^{(i)}(\Xi_n\cap W_\varrho)\Big)^2 = \lambda\int_{\mathcal{P}^0}\int_{\mathbf{R}^d}\mathbb{E}\big(J_0^{(i)}(C\cap(W_\varrho-x))\big)^2\,dx\,P^0(dC)$$

for each $i=1,\ldots,m$. By Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbf{R}^d}\big(J_0^{(i)}(C\cap(W_\varrho-x))\big)^2\,dx \quad &= \quad \int_C\int_C |(W_\varrho-y)\cap(W_\varrho-z)|\,J_0^{(i)}(dy)\,J_0^{(i)}(dz)\\ &\leq \quad |W_\varrho|\,\big(J_0^{(i)}(C)\big)^2\end{aligned} \tag{3.7}$$

and therefore, by means of (3.2),

$$\mathbb{E}\sum_{n\geq 1}\Big(J_n^{(i)}(\Xi_n\cap W_\varrho)\Big)^2 \leq \lambda\,|W_\varrho|\int_{\mathcal{P}^0}\mathbb{E}\,\big(J_0^{(i)}(C)\big)^2\,P^0(dC) < \infty\,,$$

for $i = 1, \ldots, m$. The existence of the other expressions is seen by the inequality of Cauchy–Schwarz, which proves (3.3). In order to show (3.4), notice that the properties of the family of convex sets $(W_\varrho)_{\varrho>0}$ entail that

$$\lim_{\varrho\to\infty} \frac{|(W_\varrho - y) \cap (W_\varrho - z)|}{|W_\varrho|} = 1 \qquad \text{for any fixed } y, z \in \mathbb{R}^d.$$

Hence, bounding the mixed second moments $\mathbb{E}\big(J_0^{(i)}(C \cap (W_\varrho - x))\, J_0^{(j)}(C \cap (W_\varrho - x))\big)$ from above, quite similar to (3.7), and using the dominated convergence theorem, we get

$$\lim_{\varrho\to\infty} \int_{\mathbb{R}^d} \frac{\mathbb{E}\left(J_0^{(i)}(C \cap (W_\varrho - x))\, J_0^{(j)}(C \cap (W_\varrho - x))\right)}{|W_\varrho|}\, dx = \mathbb{E}\left(J_0^{(i)}(C)\, J_0^{(j)}(C)\right)$$

for any $C \in \mathcal{P}^0$ and $i, j = 1, ..., m$. The previous relation remains true for the first–order moments resulting from the covariance formula (3.6). Thus, applying Campbell's and Fubini's theorems to (3.6), together with the dominated convergence theorem, we finally obtain that

$$\lim_{\varrho\to\infty} \frac{\operatorname{Cov}\left(Z_\varrho^{(i)}, Z_\varrho^{(j)}\right)}{|W_\varrho|}$$
$$= \lambda \int_{\mathcal{P}^0} \lim_{\varrho\to\infty} \int_{\mathbb{R}^d} \frac{\operatorname{Cov}\big(J_0^{(i)}(C \cap (W_\varrho - x))\, , J_0^{(j)}(C \cap (W_\varrho - x))\big)}{|W_\varrho|}\, dx\, P^0(dC)$$

for any $i, j = 1, \ldots, m$, where the expression on right–hand side coincides with σ_{ij}^2 as defined in (3.4).

Note that the second part of Theorem 3.1 implies that the asymptotic variance of the scalar product $t^\top \widetilde{Z}_\varrho = \sum_{l=1}^m t_l \widetilde{Z}_\varrho^{(l)}$ exists for any $t = (t_1, \ldots, t_m)^\top \in \mathbb{R}^m$, more precisely,

$$\lim_{\varrho\to\infty} \operatorname{Var}\left(t^\top \widetilde{Z}_\varrho\right) = t^\top K t = \sum_{i,j=1}^m t_i\, t_j\, \sigma_{ij}^2\ , \tag{3.8}$$

where $K = \left(\sigma_{ij}^2\right)_{i,j=1}^m$ is given by (3.4).

We conclude this section with a discussion of the integrability conditions in (3.2). Our aim is to put separate conditions on the random measures $J_0^{(i)}$ and the typical cell Ξ^* of X which together imply (3.2).

Lemma 3.1. *The following inequalities*

$$\big(\lambda^{(i)}\big)^2\, \mathbb{E}|\,\Xi^*|^2 \le \int_{\mathcal{P}^0} \mathbb{E}\big(J_0^{(i)}(C)\big)^2\, P^0(dC) \le \mathbb{E}|\,\Xi^* \oplus b(o, \sqrt{d})|^2\, \mathbb{E}\big(J_0^{(i)}([0,1)^d)\big)^2 \tag{3.9}$$

hold for each $i = 1, \ldots, m$. *Consequently,* (3.2) *is satisfied whenever*

$$\mathbb{E}\, M_k^2(\,\Xi^*) < \infty \qquad \text{and} \qquad \mathbb{E}\big(J_0^{(i)}([0,1)^d)\big)^2 < \infty \tag{3.10}$$

for any $k = 0, \ldots, d-1$ *and* $i = 1, \ldots, m$.

Proof. Taking into account that $\mathbb{E}J_0^{(i)}(C) = \lambda^{(i)}|C|$, the first inequality in (3.9) immediately follows from Jensen's inequality $\big(\mathbb{E}J_0^{(i)}(C)\big)^2 \le \mathbb{E}\big((J_0^{(i)}(C))^2\big)$. The obvious set–theoretic inclusions

$$C \subseteq \bigcup_{z\in\mathbb{Z}^d : C\cap([0,1)^d+z)\neq\emptyset} \big([0,1)^d + z\big) \subseteq C \oplus b(o,\sqrt{d}) ,$$

which are true for any subset C of $\mathbb{R}^d$, and Steiner's formula (2.1) imply that

$$\#\{z\in\mathbb{Z}^d : C\cap([0,1)^d+z)\neq\emptyset\} \le |C\oplus b(o,\sqrt{d})| = \sum_{k=0}^{d}\binom{d}{k} M_k(C)\, d^{k/2} \tag{3.11}$$

for each $C\in\mathcal{C}$. Thus, using the monotonicity and translation–invariance of the set function $\mathbb{E}\big(J_0^{(i)}(\cdot)\big)^2$ and the elementary inequality $(a_1+\cdots+a_N)^2 \le N\,(a_1^2+\cdots+a_N^2)$, we arrive at

$$\begin{aligned}\mathbb{E}\big(J_0^{(i)}(C)\big)^2 &\le \#\{z\in\mathbb{Z}_C^d\}\sum_{z\in\mathbb{Z}_C^d}\mathbb{E}\big(J_0^{(i)}([0,1)^d+z)\big)^2\\ &\le |C\oplus b(o,\sqrt{d})|^2\,\mathbb{E}\big(J_0^{(i)}([0,1)^d)\big)^2 ,\end{aligned}$$

where $\mathbb{Z}_C^d = \{z\in\mathbb{Z}^d : C\cap([0,1)^d+z)\neq\emptyset\}$. This proves the second inequality in (3.9). Furthermore, by (3.11), $\mathbb{E}|\,\Xi^*\oplus b(o,\sqrt{d})|^2 < \infty$ is true if the first integrability condition in (3.10) is satisfied.

In some cases, it might not be possible to directly check whether or not the first integrability condition in (3.10) is satisfied. This is due to the fact that it is sometimes difficult to determine the second moment $\mathbb{E}\,M_k^2(\,\Xi^*)$ of the k-th Minkowski–functional $M_k(\,\Xi^*)$ of the typical cell Ξ^* of X. However, the isodiametric inequality (2.2) implies that $\mathbb{E}\,M_k^2(\,\Xi^*) < \infty$ holds for each $k = 0,\ldots,d-1$ provided that

$$\mathbb{E}D^{2d}(\Xi^*) < \infty . \tag{3.12}$$

4. Laws of large numbers

Recall that the individual ergodic theorem applied to the (stationary and ergodic) marked point process $\sum_{n\ge1}\delta_{[\alpha(\Xi_n),\Xi_n^0]}$ reads as follows; see [6], p.339. For any real–valued integrable function $h\in L^1(\mathcal{P}^0,\mathcal{B}(\mathcal{F})\cap\mathcal{P}^0,P^0)$, we have

$$\frac{1}{|W_\varrho|}\sum_{n\ge1}\mathbb{I}_{W_\varrho}(\alpha(\Xi_n))\,h(\Xi_n^0) \xrightarrow[\varrho\to\infty]{\text{a.s.}} \lambda\,\mathbb{E}h(\Xi^*) = \lambda\int_{\mathcal{P}^0} h(C)P^0(dC) . \tag{4.1}$$

However, in the context of this paper as in many other statistical applications of the spatial ergodic theorem (4.1), we have to consider spatial averages over cells of $X = \{\Xi_n\}_{n\ge1}$ which belong to the sampling window W_ϱ only partly. Such boundary effects are being taken into account by the subsequent result, which provides the strong consistency of a (not necessarily unbiased) estimator for $\lambda\,\mathbb{E}g(\Xi^*)$ in the case of a non–random, translation–invariant, and isotonic functional on $\mathcal{C}$.

Theorem 4.1. *Let $g : \mathcal{C} \to [0,\infty)$ be a $\mathcal{B}(\mathcal{F})$–measurable, non–negative set–function such that $g(C) \le g(C')$ for $C \subseteq C'$ and $g(C) = g(C+x)$ for any $C \in \mathcal{C}$ and $x \in R^d$. If the typical cell Ξ^* of X satisfies*

$$\mathbb{E}D^d(\Xi^*) < \infty \qquad and \qquad \mathbb{E}g(\Xi^*) < \infty\,, \tag{4.2}$$

then

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} \mathbb{I}_{\{\Xi_n\cap W_\varrho\neq\emptyset\}}\, g(\Xi_n \cap W_\varrho) \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda\,\mathbb{E}g(\Xi^*)\,. \tag{4.3}$$

The proof of Theorem 4.1 is postponed to Section 4.2. Notice however that Theorem 4.1 is not completely new. For instance, in the planar case $d = 2$, one can find it in [5] applied to some particular functionals $g(\Xi_n \cap W_\varrho)$ of the cells $\Xi_n \cap W_\varrho$, whereas we consider a general class of isotonic and translation–invariant functionals $g : \mathcal{C} \to [0,\infty)$. Furthermore, a straightforward application of Theorem 4.1 to each of the particular functionals

$$g_1(C) = \mathbb{E}\Big(J_0^{(i)}(C)\, J_0^{(j)}(C)\Big) \qquad \text{and} \qquad g_2(C) = |C|^2\,, \qquad 1 \le i,j \le m,\ C \in \mathcal{C},$$

yields the following result.

Corollary 4.1. *Let $J_0 = (J_0^{(1)},\ldots,J_0^{(m)})^\top$ be a vector of stationary random measures on $\mathbb{R}^d$ being independent of X and satisfying both* (3.2) (*or* (3.10)) *and* $\mathbb{E}D^d(\Xi^*) < \infty$. *Then, for any $t = (t_1,\ldots,t_m)^\top \in \mathbb{R}^m$,*

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} \mathbb{E}_X\big(t^\top J_0(\Xi_n\cap W_\varrho)\big)^2 \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda\,\mathbb{E}\big(t^\top J_0(\Xi^*)\big)^2 \tag{4.4}$$

and

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} |\Xi_n\cap W_\varrho|^2 \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda\,\mathbb{E}|\,\Xi^*|^2\,. \tag{4.5}$$

It turns out that the result of Theorem 4.1 remains true if the (non–random) functional $g : \mathcal{C} \to [0,\infty)$ is replaced by the stationary random measures $J_n^{(i)}$.

Theorem 4.2. *Assume that $\mathbb{E}D^d(\Xi^*) < \infty$ and $\lambda^{(i)} = \mathbb{E}J_0^{(i)}([0,1)^d) < \infty$ for $i = 1,\ldots,m$. Then*

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} J_n^{(i)}(\Xi_n\cap W_\varrho) \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda^{(i)}\,, \qquad i = 1,\ldots,m\,. \tag{4.6}$$

The proof of Theorem 4.2 will be given in Section 4.3 below. Notice that the limit $\lambda^{(i)}$ in (4.6) corresponds to that of (4.3), because the stationarity of $J_0^{(i)}$ and (2.4) imply that $\lambda^{(i)} = \lambda\,\mathbb{E}\,J_0^{(i)}(\Xi^*)$ provided that J_0 and Ξ^* are independent.

4.1. Cells hitting the boundary of the sampling window

The following lemma is essential for the proof of Theorem 4.1. However, it also seems to be of interest in its own right. We show that the influence of those cells Ξ_n of X on the left–hand side of (4.3) hitting the boundary ∂W_ϱ becomes asymptotically neglegible as $\varrho \to \infty$. For this to show we define the families of events $(A_\varrho)_{\varrho>\varrho_0}$ and $(B_\varrho)_{\varrho>\varrho_0}$ by

$$A_\varrho = \bigcap_{n\geq 1}\Big(\big\{\, (\Xi_n^0 + \alpha(\Xi_n)) \cap W_\varrho = \emptyset \,\big\} \cup \big\{\alpha(\Xi_n) \in W_{\varrho+q(\varrho)}\big\}\Big) \tag{4.7}$$

and

$$B_\varrho = \bigcap_{n\geq 1}\Big(\big\{\, (\Xi_n^0 + \alpha(\Xi_n)) \subseteq W_\varrho \,\big\} \cup \big\{\alpha(\Xi_n) \notin W_{\varrho-q(\varrho)}\big\}\Big)\,, \tag{4.8}$$

where $\varrho_0 \geq 0$ is some constant and the function $q : (\varrho_0, \infty) \to (0,\infty)$ is such that $q(\varrho) < \varrho$ for each $\varrho > \varrho_0$.

Lemma 4.1. *Under the assumption that* $\mathbb{E}D^d(\Xi^*) < \infty$, *there exists a non–decreasing function* $q : (\varrho_0, \infty) \to (0,\infty)$ *satisfying* $q(\varrho) < \varrho$ *for* $\varrho > \varrho_0$, $q(\varrho) \to \infty$, *and* $\frac{q(\varrho)}{\varrho} \downarrow 0$ *as* $\varrho \to \infty$ *such that*

$$\lim_{\ell\to\infty} \mathbb{P}\Big(\bigcup_{k\geq \ell} A_k^c\Big) = 0 \qquad and \qquad \lim_{\ell\to\infty} \mathbb{P}\Big(\bigcup_{k\geq \ell} B_k^c\Big) = 0\,. \tag{4.9}$$

Proof. To begin with we recall the well–known fact from analysis that the integrability of $D^d(\Xi^*)$ implies the existence of a convex function $H : [0,\infty) \to [0,\infty)$ strictly increasing on its support (x_0, ∞) (for some $x_0 \geq 0$) such that $H(x)/x$ is non-decreasing for $x > 0$ with $\lim_{x\to\infty} H(x)/x = \infty$ and $\mathbb{E}H(D^d(\Xi^*)) < \infty$; see e.g. Theorem II.22 in [7]. Furthermore, for some $r > 0$ such that $b(o,r) \subseteq W$, the value $q(\varrho)$ $(> x_0^{1/d}/r)$ is defined as the unique solution of the equation

$$\varrho^d \;=\; H(r^d\, q^d(\varrho)) \qquad \text{for any} \qquad \varrho > 0\,. \tag{4.10}$$

It is easily checked that the function $\varrho \mapsto q(\varrho)$ possesses the required properties for $\varrho > \varrho_0 = \inf\{x > 0 : H(r^d q^d(x)) > q^d(x)\}$. Note that $b(o,r) \subseteq W$ implies that $b(o, r\,q(\varrho)) \subseteq W_{q(\varrho)}$ and, by the convexity of W,

$$W_\varrho \oplus b(o, rq(\varrho)) \subseteq W_\varrho \oplus W_{q(\varrho)} \subseteq W_{\varrho+q(\varrho)}\,.$$

Thus, for any $C \in \mathcal{C}$ with $o \in C$, we have $D(C) > r\,q(\varrho)$ provided that $W^c_{\varrho+q(\varrho)} \cap (W_\varrho \oplus C) \neq \emptyset$. By the definition of A_ϱ and the latter implication, this yields that

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq\ell} A_k^c\Big) &= \mathbb{P}\Big(\bigcup_{n\geq 1}\bigcup_{k\geq \ell}\{\alpha(\Xi_n) \in W^c_{k+q(k)} \cap (W_k \oplus \check{\Xi}_n^0)\}\Big)\\
&\leq \mathbb{P}\Big(\bigcup_{n\geq 1}\bigcup_{k\geq \ell}\Big(\big\{\alpha(\Xi_n) \in W_k \oplus b(o, D(\Xi_n^0))\big\} \cap \big\{D(\Xi_n^0) > r\,q(k)\big\}\Big)\Big)\\
&= \mathbb{P}\Big(\bigcup_{n\geq 1}\bigcup_{k\geq \ell}\Big(\big\{\alpha(\Xi_n) \in W_k \oplus b(o, D(\Xi_n^0))\big\}\\
&\qquad\qquad \cap \big\{r\,q(k) < D(\Xi_n^0) \leq r\,q(k+1)\big\}\Big)\Big)\,.
\end{aligned} \tag{4.11}$$

The last equality becomes clear by the fact that for any two sequences of events $\{E_k\}_{k\geq\ell}$ and $\{E'_k\}_{k\geq\ell}$ with $E_k \subseteq E_{k+1}$ and $E'_k \supseteq E'_{k+1}$ for $k \geq \ell$, the identity

$$\bigcup_{k\geq\ell} E_k \cap E'_k = \bigcup_{k\geq\ell} E_k \cap (E'_k \setminus E'_{k+1})$$

holds. Thus, the subadditivity of $\mathbb{P}$, the inequality $\mathbb{P}(U \geq 1) \leq \mathbb{E}|U|$ and Campbell's theorem for stationary marked point processes imply that

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq\ell} A_k^c\Big) &\leq \sum_{k\geq\ell} \mathbb{P}\Big(\sum_{n\geq1} \mathbb{1}_{W_k\oplus b(o,D(\Xi_n^0))}(\alpha(\Xi_n))\, \mathbb{1}_{(r\,q(k),\,r\,q(k+1)]}(D(\Xi_n^0)) \geq 1\Big) \\
&\leq \sum_{k\geq\ell} \mathbb{E}\Big(\sum_{n\geq1} \mathbb{1}_{W_k\oplus b(o,D(\Xi_n^0))}(\alpha(\Xi_n))\, \mathbb{1}_{(r\,q(k),\,r\,q(k+1)]}(D(\Xi_n^0))\Big) \\
&= \lambda \sum_{k\geq\ell} \int_{\mathcal{P}^0} \big|W_k \oplus b(o, D(C))\big|\, \mathbb{1}_{(r\,q(k),\,r\,q(k+1)]}(D(C))\, P^0(dC)\,.
\end{aligned}$$

Since $W_k = k\,W \in \mathcal{C}$, we now are in a position to apply Steiner's formula (2.1) which together with the homogeneity relation $M_s(W_k) = k^{d-s} M_s(W)$ for the Minkowski–functionals M_s (see e.g. [12]) and the monotonicity of the function $H(\cdot)$ leads to

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq\ell} A_k^c\Big) &\leq \lambda \sum_{k\geq\ell} \sum_{s=0}^{d} \binom{d}{s} k^{d-s} M_s(W)\, \mathbb{E}\Big(D^s(\Xi^*)\, \mathbb{1}_{(r\,q(k),\,r\,q(k+1)]}(D(\Xi^*))\Big) \\
&\leq \lambda \sum_{k\geq\ell} \sum_{s=0}^{d} \binom{d}{s} M_s(W)\, \frac{k^{d-s}\, r^s\, q^s(k+1)}{H(r^d\, q^d(k))} \\
&\qquad \times\, \mathbb{E}\Big(H(D^d(\Xi^*))\, \mathbb{1}_{(r\,q(k),\,r\,q(k+1)]}(D(\Xi^*))\Big)\,.
\end{aligned}$$

By (4.10), we have $H(r^d\, q^d(k)) = k^d$ and $q(k+1) \leq k+1$ for all $k \geq \ell$ (ℓ large enough) and therefore,

$$\frac{k^{d-s}\, r^s\, q^s(k+1)}{H(r^d\, q^d(k))} \leq \Big(1 + \frac{1}{k}\Big)^s r^s \leq 2^s\, r^s \quad \text{for} \quad k \geq \ell\,.$$

This yields

$$\mathbb{P}\big(\bigcup_{k\geq\ell} A_k^c\big) \leq \lambda \sum_{s=0}^{d} \binom{d}{s} (2\,r)^s\, M_s(W)\, \mathbb{E}\big(H(D^d(\Xi^*))\, \mathbb{1}_{(r\,q(\ell),\infty)}(D(\Xi^*))\big) \underset{\ell\to\infty}{\longrightarrow} 0\,.$$

To show the second assertion in (4.9) we first realize that

$$\mathbb{P}\big(\bigcup_{k\geq\ell} B_k^c\big) = \mathbb{P}\Big(\bigcup_{n\geq1} \bigcup_{k\geq\ell} \{\alpha(\Xi_n) \in W_{k-q(k)} \cap (W_k^c \oplus \check{\Xi}_n^0)\}\Big)\,.$$

In analogy to the considerations above one can show that, for any $C \in \mathcal{C}$ with $o \in C$, we have $D(C) > rq(\varrho)$ provided that $\big(W_\varrho^c \oplus C\big) \cap W_{\varrho-q(\varrho)} \neq \emptyset$. Together with $W_{k-q(k)} \subseteq W_k$, this gives that

$$\mathbb{P}\big(\bigcup_{k\geq\ell} B_k^c\big) \leq \mathbb{P}\Big(\bigcup_{n\geq1} \bigcup_{k\geq\ell} \Big(\{\alpha(\Xi_n) \in W_k\} \cap \{D(\Xi_n^0) > r\,q(k)\}\Big)\Big)\,.$$

Since the right–hand side of the latter inequality is not larger than the bound on the right–hand side of (4.11), we immediately get that $\mathbb{P}\Big(\bigcup_{k\geq\ell} B_k^c\Big) \underset{\ell\to\infty}{\longrightarrow} 0$.

4.2. Proof of Theorem 4.1

For notational simplicity we put

$$\Delta_k = \frac{1}{|W_k|}\sum_{n\geq 1} \mathbb{I}_{\{(\Xi_n^0+\alpha(\Xi_n))\cap W_k\neq\emptyset\}}\, g((\Xi_n^0+\alpha(\Xi_n))\cap W_k) \quad \text{for} \quad k\geq\ell\,.$$

By rewriting the almost sure convergence, the assertion of Theorem 4.1 is equivalent to

$$\mathbb{P}\Big(\sup_{k\geq\ell}\big|\Delta_k-\lambda\,\mathbb{E}g(\Xi^*)\big|\geq\delta\Big)=\mathbb{P}\Big(\bigcup_{k\geq\ell}\big\{\big|\Delta_k-\lambda\,\mathbb{E}g(\Xi^*)\big|\geq\delta\big\}\Big)\underset{\ell\to\infty}{\longrightarrow} 0 \tag{4.12}$$

for any $\delta>0$; see e.g. Lemma 6.8 in [20]. Furthermore,

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{k\geq\ell}\big\{\,\big|\Delta_k-\lambda\,\mathbb{E}g(\Xi^*)\,\big|\geq\delta\big\}\Big) &\leq \mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\big|\Delta_k-\lambda\,\mathbb{E}g(\Xi^*)\,\big|\geq\delta\big\}\cap A_k\cap B_k\Big)\Big)\\
&+\ \mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\big|\Delta_k-\lambda\,\mathbb{E}g(\Xi^*)\,\big|\geq\delta\big\}\cap (A_k\cap B_k)^c\Big)\Big)\\
&\leq\ \mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\Delta_k\geq\lambda\,\mathbb{E}g(\Xi^*)+\delta\big\}\cap A_k\Big)\Big)+\mathbb{P}\Big(\bigcup_{k\geq\ell}A_k^c\Big)\\
&+\ \mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\Delta_k\leq\lambda\,\mathbb{E}g(\Xi^*)-\delta\big\}\cap B_k\Big)\Big)+\mathbb{P}\Big(\bigcup_{k\geq\ell}B_k^c\Big)\,,
\end{aligned}$$

where the events A_k and B_k have been defined in (4.7) and (4.8), respectively. Taking into account the properties of the functional $g:\mathcal{C}\to[0,\infty)$, on A_k we can verify the inequality

$$\Delta_k\leq\frac{1}{|W_k|}\sum_{n\geq 1}\mathbb{I}_{W_{k+q(k)}}(\alpha(\Xi_n))\,g(\Xi_n^0)$$

for each $k\geq\ell$ with ℓ sufficiently large. Likewise, for $k\geq\ell$, we have on B_k that

$$\Delta_k\geq\frac{1}{|W_k|}\sum_{n\geq 1}\mathbb{I}_{W_{k-q(k)}}(\alpha(\Xi_n))\,g(\Xi_n^0)\,.$$

Hence,

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\Delta_k\geq\lambda_0\,\mathbb{E}g(\Xi^*)+\delta\big\}\cap A_k\Big)\Big)+\mathbb{P}\Big(\bigcup_{k\geq\ell}\Big(\big\{\,\Delta_k\leq\lambda\,\mathbb{E}g(\Xi^*)-\delta\big\}\cap B_k\Big)\Big)\\
&\leq\sum_{\kappa\in\{-1,+1\}}\mathbb{P}\Big(\sup_{k\geq\ell}\Big|\frac{1}{|W_k|}\sum_{n\geq 1}\mathbb{I}_{W_{k+\kappa\,q(k)}}(\alpha(\Xi_n))\,g(\Xi_n^0)-\lambda\,\mathbb{E}g(\Xi^*)\,\Big|\geq\delta\Big)\,,
\end{aligned}$$

However, the latter sum tends to zero as $\ell\longrightarrow\infty$, since the spatial ergodic theorem (4.1) yields

$$\frac{1}{|W_{\varrho+\kappa\,q(\varrho)}|}\sum_{n\geq 1}\mathbb{I}_{W_{\varrho+\kappa\,q(k)}}(\alpha(\Xi_n))\,g(\Xi_n^0)\ \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}}\ \lambda\,\mathbb{E}g(\Xi^*)$$

and the asymptotic behaviour of $q(\varrho)$ as $\varrho \to \infty$ implies that

$$\frac{|W_{\varrho+\kappa\, q(\varrho)}|}{|W_\varrho|} = \left(1 + \kappa\,\frac{q(\varrho)}{\varrho}\right)^d \underset{\varrho\to\infty}{\longrightarrow} 1$$

for $\kappa = \pm 1$. Finally, Lemma 4.1 shows the validity of (4.12).

4.3. Proof of Theorem 4.2

By Lemma 4.1 and the arguments of the foregoing proof of Theorem 4.1, one can show that (4.6) is equivalent to

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} \mathbb{1}_{W_\varrho}(\alpha(\Xi_n))\, J_n^{(i)}(\Xi_n) \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda^{(i)} \qquad \text{for } i = 1,\dots,m.$$

Furthermore, a simple application of (4.1) to the function $h(C) = |C|$ for $C \in \mathcal{P}^0$ yields

$$\frac{1}{|W_\varrho|}\sum_{n\ge 1} \mathbb{1}_{W_\varrho}(\alpha(\Xi_n))\, |\Xi_n| \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda\,\mathbb{E}|\Xi^*| = 1\,,$$

where the latter equality follows from (2.4). Thus, (4.6) is equivalent to $S_N/|W_N| \underset{N\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} 0$, where S_N can be written as partial sum $S_N = U_1 + \cdots + U_N$ with

$$U_k = \sum_{n\ge 1} \mathbb{1}_{W_k\setminus W_{k-1}}(\alpha(\Xi_n))\,\big(J_n^{(i)}(\Xi_n) - \lambda^{(i)}\,|\Xi_n|\big)\,, \qquad k = 1,2,\dots\,.$$

Here and below the index $i = 1,\dots,m$ is fixed. In order to prove that $S_N/|W_N| \underset{N\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} 0$ holds, it is necessary and sufficient to show that for any given $\delta, \eta > 0$, there exists an integer $\ell_0 = \ell_0(\delta,\eta)$ such that

$$\mathbb{P}\Big(\sup_{k\ge \ell}\frac{|S_k|}{|W_k|} \ge \delta\Big) \le \eta \qquad \text{for any } \ell \ge \ell_0. \tag{4.13}$$

For $\varepsilon > 0$ (below chosen as function of δ and η) and $a_k = |W_k|$, we introduce the truncated random variables

$$U_k^{(\varepsilon)} = \sum_{n\ge 1} \mathbb{1}_{W_k\setminus W_{k-1}}(\alpha(\Xi_n))\,\big(J_n^{(i)}(\Xi_n) - \lambda^{(i)}\,|\Xi_n|\big)\,\mathbb{1}_{\{|J_n^{(i)}(\Xi_n) - \lambda^{(i)}\,|\Xi_n||\le\varepsilon\,(a_k\vee a_\ell)\}}$$

and their partial sums $S_k^{(\varepsilon)} = U_1^{(\varepsilon)} + \cdots + U_k^{(\varepsilon)}$ for $k \ge 1$. Note that the random variables $U_k^{(\varepsilon)}$ are conditionally independent given the tessellation $X = \{\Xi_n\}_{n\ge 1}$. Since $S_k^{(\varepsilon)}(\omega) = S_k(\omega)$ for any $k \ge 1$, whenever

$$\omega \in A_{\varepsilon,\ell}(X) = \bigcap_{k\ge 1}\ \bigcap_{n:\alpha(\Xi_n)\in W_k\setminus W_{k-1}} \big\{\big|J_n^{(i)}(\Xi_n) - \lambda^{(i)}\,|\Xi_n|\big| \le \varepsilon\,(a_k \vee a_\ell)\big\}\,,$$

it follows by a standard estimate that, for any fixed $\delta > 0$,

$$\left|\mathbb{P}\Big(\sup_{k\ge\ell}\frac{|S_k|}{a_k} \ge \delta\Big) - \mathbb{P}\Big(\sup_{k\ge\ell}\frac{|S_k^{(\varepsilon)}|}{a_k} \ge \delta\Big)\right| \le \mathbb{P}\big(A^c_{\varepsilon,\ell}(X)\big)\,. \tag{4.14}$$

Using Campbell's theorem, a straightforward computation reveals that

$$\begin{aligned}\mathbb{P}\big(A^c_{\varepsilon,\ell}(X)\big) &\le \sum_{k\ge 1}\mathbb{P}\Big(\sum_{n\ge 1}\mathbb{I}_{W_k\setminus W_{k-1}}(\alpha(\Xi_n))\,\mathbb{I}_{\{|\,J_n^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|\,|>\varepsilon\,(a_k\vee a_\ell)\}}\ge 1\Big)\\ &\le \lambda\, a_\ell\,\mathbb{P}\big(V>\varepsilon\, a_\ell\big)+\lambda\sum_{k>\ell}(a_k-a_{k-1})\,\mathbb{P}\big(V>\varepsilon\, a_k\big)\le\ \frac{\lambda}{\varepsilon}\,\mathbb{E}\big(V\,\mathbb{I}_{\{V>\varepsilon\, a_\ell\}}\big)\,,\end{aligned}$$

where, for fixed i, V denotes the non–negative random variable $\big|J_0^{(i)}(\Xi^*)-\lambda^{(i)}\,|\Xi^*|\big|$ with $\mathbb{E}V<\infty$. Thus, $\mathbb{P}\big(A^c_{\varepsilon,\ell}(X)\big)\to 0$ as $\ell\to\infty$. Furthermore, we have

$$\begin{aligned}\mathbb{P}\Big(\sup_{k\ge\ell}\frac{|S_k^{(\varepsilon)}|}{a_k}\ge\delta\Big)\le\mathbb{E}\Big(\mathbb{P}_X\Big(\sup_{k\ge\ell}\frac{|\,S_k^{(\varepsilon)}-\mathbb{E}_X S_k^{(\varepsilon)}\,|}{a_k}\ge\frac{\delta}{2}\Big)\Big)&\\ +\,\mathbb{E}\left(\mathbb{P}_X\Big(\sup_{k\ge\ell}\frac{|\mathbb{E}_X S_k^{(\varepsilon)}|}{a_k}\ge\frac{\delta}{2}\Big)\right)&\,,\end{aligned}\tag{4.15}$$

where $\mathbb{P}_X$ denotes the conditional probability given the tessellation X. To estimate the first term on the right-hand side of (4.15), we make use of the well-known Hàjek–Rényi inequality, which reads as follows; see e.g. Theorem 2.5 in [20]. For (conditionally) independent mean zero random variables $V_1,V_2,\ldots$ with finite variances and positive constants $c_1,c_2,\ldots$ satisfying $c_1\ge c_2\ge\cdots$, the inequality

$$\mathbb{P}_X\Big(\max_{\ell\le k\le L}c_k\,|V_1+\cdots+V_k|\ge x\Big)\le\frac{1}{x^2}\left(c_\ell^2\sum_{k=1}^{\ell}\mathbb{E}_X V_k^2+\sum_{k=\ell+1}^{L}c_k^2\,\mathbb{E}_X V_k^2\right)$$

holds for any $x>0$ and $1\le\ell\le L$. Applying this inequality to the conditionally independent random variables $V_k=U_k^{(\varepsilon)}-\mathbb{E}_X U_k^{(\varepsilon)}$ (having conditional mean zero) with $c_k=1/a_k$ for $k\ge\ell$, gives

$$\mathbb{P}\Big(\sup_{k\ge\ell}\frac{|S_k^{(\varepsilon)}-\mathbb{E}_X S_k^{(\varepsilon)}|}{a_k}\ge\frac{\delta}{2}\Big)\le\frac{4}{\delta^2}\,\frac{\mathbb{E}\mathrm{Var}_X(S_\ell^{(\varepsilon)})}{a_\ell^2}+\frac{4}{\delta^2}\sum_{k>\ell}\frac{\mathbb{E}\mathrm{Var}_X(U_k^{(\varepsilon)})}{a_k^2}\,.\tag{4.16}$$

Having in mind that, given the tessellation $X=\{\Xi_n\}_{n\ge 1}$, the random variables $J_n^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|$, $n\ge 1$, are mutually independent with mean zero, it is clear that

$$\mathrm{Var}_X(S_\ell^{(\varepsilon)})\le\sum_{n\ge 1}\mathbb{I}_{W_\ell}(\alpha(\Xi_n))\,\mathbb{E}_X\Big(\big(J_0^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|\big)^2\,\mathbb{I}_{\{|\,J_0^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|\,|\le\varepsilon\, a_\ell\}}\Big)\,.$$

and, for $k>\ell$,

$$\begin{aligned}&\mathrm{Var}_X(U_k^{(\varepsilon)})\\ &\quad\le\sum_{n\ge 1}\mathbb{I}_{W_k\setminus W_{k-1}}(\alpha(\Xi_n))\,\mathbb{E}_X\Big(\big(J_0^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|\big)^2\,\mathbb{I}_{\{|\,J_0^{(i)}(\Xi_n)-\lambda^{(i)}\,|\Xi_n|\,|\le\varepsilon\, a_k\}}\Big)\,.\end{aligned}$$

Applying Campbell's theorem again and using that $a_k \le a_{k+1}$ and $a_k^2 \ge a_k\, a_{k-1}$ for $k \ge 1$, we find (after a series of elementary rearrangements) that

$$
\begin{aligned}
&\frac{\mathbb{E}\mathrm{Var}_X(S_\ell^{(\varepsilon)})}{a_\ell^2} + \sum_{k>\ell} \frac{\mathbb{E}\mathrm{Var}_X(U_k^{(\varepsilon)})}{a_k^2} \\
&\quad \le \frac{\lambda}{a_\ell}\, \mathbb{E}\big(V^2\, \mathbb{1}_{\{V \le \varepsilon\, a_\ell\}}\big) + \lambda \sum_{k>\ell} \frac{a_k - a_{k-1}}{a_{k-1}\, a_k}\, \mathbb{E}\big(V^2\, \mathbb{1}_{\{V \le \varepsilon\, a_k\}}\big) \\
&\quad \le \lambda\, \varepsilon\, \mathbb{E}V + \lambda\, \varepsilon\, \frac{a_{\ell+1}}{a_\ell} \mathbb{E}\big(V\, \mathbb{1}_{\{V \le \varepsilon\, a_{\ell+1}\}}\big) + \lambda\, \varepsilon \sum_{k>\ell} \frac{a_{k+1}}{a_k} \mathbb{E}\big(V\, \mathbb{1}_{\{\varepsilon\, a_k < V \le \varepsilon\, a_{k+1}\}}\big) \\
&\quad \le \lambda\, \varepsilon \Big(1 + \frac{a_{\ell+1}}{a_\ell}\Big)\, \mathbb{E}V \le \frac{\eta\, \delta^2}{12}
\end{aligned}
$$

for any $\ell \ge 1$ provided that we put $\varepsilon = \eta\, \delta^2/12(1+2^d)\, \lambda\, \mathbb{E}V$. Thus, the right–hand side of (4.16) does not exceed $\eta/3$ for $\ell \ge 1$. Further, since $\mathbb{E}_X\big(\, J_n^{(i)}(\Xi_n) - \lambda^{(i)}\, |\Xi_n|\,\big) = 0$, one can easily show that $\big|\, \mathbb{E}_X S_k^{(\varepsilon)}\, \big| \le Z_{k,\ell}^{(\varepsilon)}$ for any $k \ge \ell$, where

$$
Z_{k,\ell}^{(\varepsilon)} = \sum_{n \ge 1} \mathbb{1}_{W_k}(\alpha(\Xi_n))\, \mathbb{E}_X \big|\, J_0^{(i)}(\Xi_n) - \lambda^{(i)}\, |\Xi_n|\, \big|\, \mathbb{1}_{\{|\, J_0^{(i)}(\Xi_n) - \lambda^{(i)}\, |\Xi_n|\, | > \varepsilon\, a_\ell\}}\, .
$$

With the above choice of $\varepsilon > 0$, take $\ell_1 = \ell_1(\delta, \eta)$ to be the smallest integer such that $\lambda\, \mathbb{E}V\, \mathbb{1}_{\{V > \varepsilon\, a_{\ell_1}\}} \le (\delta \wedge \eta)/3$. This implies that $\max_{\ell \ge \ell_1} \mathbb{P}\big(A^c_{\varepsilon,\ell}(X)\big) \le \eta/3$ and, for $\ell \ge \ell_1$,

$$
\begin{aligned}
\mathbb{P}\Big(\sup_{k \ge \ell} \frac{|\mathbb{E}_X S_k^{(\varepsilon)}|}{a_k} \ge \frac{\delta}{2}\Big) &\le \mathbb{P}\Big(\sup_{k \ge \ell} \frac{Z_{k,\ell_1}^{(\varepsilon)}}{a_k} \ge \frac{\delta}{2}\Big) \\
&\le \mathbb{P}\Big(\sup_{k \ge \ell} \frac{|Z_{k,\ell_1}^{(\varepsilon)} - a_k\, \lambda\, \mathbb{E}V\, \mathbb{1}_{\{V > \varepsilon\, a_{\ell_1}\}}|}{a_k} \ge \frac{\delta}{6}\Big)\, .
\end{aligned}
\tag{4.17}
$$

However, since the ergodic theorem (4.1) yields $\big(Z_{N,\ell_1}^{(\varepsilon)}/a_N\big) \xrightarrow{\text{a.s.}} \lambda\, \mathbb{E}V\, \mathbb{1}_{\{V > \varepsilon\, a_{\ell_1}\}}$ as $N \to \infty$, there exits an integer $\ell_0 = \ell_0(\delta, \eta)$ (larger than ℓ_1) such that the right-hand side of (4.17) becomes smaller than $\eta/3$ for any $\ell \ge \ell_0$. Together with the other estimates above and combined with (4.14)–(4.16), this yields (4.13).

5. Multivariate central limit theorem

In this section we prove a central limit theorem, which states asymptotic normality of the normalized random vector $\widetilde{Z}_\varrho = \big(\widetilde{Z}_\varrho^{(1)}, \ldots, \widetilde{Z}_\varrho^{(m)}\big)^\top$ defined in (1.2) as $\varrho \to \infty$.

Theorem 5.1. *Suppose that the conditions* (3.2) (*or* (3.10)) *and* $\mathbb{E}D^d(\Xi^*) < \infty$ *are satisfied. Further, assume that the asymptotic covariance matrix* $K = \big(\sigma_{ij}^2\big)_{i,j=1}^m$ *is distinct from the null matrix, i.e.* $\max_{1 \le i \le m} \sigma_{ii}^2 > 0$. *Then,*

$$
\lim_{\varrho \to \infty} \sup_{x \in \mathbf{R}^m} \big|\mathbb{P}\big(\widetilde{Z}_\varrho \le x\big) - \Phi_K(x)\big| = 0\, , \tag{5.1}
$$

where Φ_K *denotes the distribution function of the (m-dimensional) mean zero Gaussian vector* $\mathrm{N}(o, K)$ *with covariance matrix* K.

To make the proof of Theorem 5.1 more transparent, we first collect some preliminary results in Section 5.1 and postpone the main part of the proof to Section 5.2 below.

In the particular case $d = 2$, the isodiametric inequality (2.2) implies that the second moment of the perimeter $M_1(\Xi^*)$ of Ξ^* exists if and only if the second moment of the diameter of Ξ^* exists. Therefore, in this case, $\mathbb{E}D^2(\Xi^*) < \infty$ and $\mathbb{E}|\Xi^*|^2 < \infty$ are the only conditions on the typical cell Ξ^* of X, which are needed to show (5.1).

5.1. Some auxiliary results

For any fixed vector $t = (t_1, \ldots, t_m)^\top \in \mathbb{R}^m$ and any $\varepsilon > 0$, define $\sigma^2(t) = t^\top K t \; (\geq 0)$ and the event

$$E_\varrho(t,\varepsilon) = \left\{ \left| B_\varrho^2(t,X) - \sigma^2(t)\,|W_\varrho| \right| < \varepsilon\,|W_\varrho| \right\},$$

where

$$B_\varrho^2(t,X) = \sum_{n\geq 1} \mathbb{E}_X \Big(\sum_{i=1}^m t_i \Big(J_0^{(i)}(\Xi_n \cap W_\varrho) - \lambda^{(i)}\,|\Xi_n \cap W_\varrho| \Big)\Big)^2 .$$

Lemma 5.1. *Under the conditions of Theorem* 5.1*, for any* $t \in \mathbb{R}^m$ *and* $\varepsilon > 0$,

$$\lim_{\varrho\to\infty} \mathbb{P}\big(E_\varrho^c(t,\varepsilon)\big) = 0 . \tag{5.2}$$

Proof. Using the fact that $\mathbb{E}_X\big(J_0^{(i)}(\Xi_n \cap W_\varrho)\big) = \lambda^{(i)}\,|\Xi_n \cap W_\varrho|$ for $i = 1, \ldots, m$, it is easy to see that the identity

$$\mathbb{E}_X\Big(\sum_{i=1}^m t_i\big(J_0^{(i)}(C_{n\varrho}) - \lambda^{(i)}\,|C_{n\varrho}|\big)\Big)^2 = \mathbb{E}_X\Big(\sum_{i=1}^m t_i J_0^{(i)}(C_{n\varrho})\Big)^2 - |C_{n\varrho}|^2\,\Big(\sum_{i=1}^m t_i\lambda^{(i)}\Big)^2$$

holds for any cell $\Xi_n \in X$, where $C_{n\varrho} = \Xi_n \cap W_\varrho$. Thus, by the relations (4.4) and (4.5) of Corollary 4.1,

$$\frac{B_\varrho^2(t,X)}{|W_\varrho|} \underset{\varrho\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \lambda\,\mathbb{E}\Big(\sum_{i=1}^m t_i J_0^{(i)}(\Xi^*)\Big)^2 - \lambda\,\Big(\sum_{i=1}^m t_i\lambda^{(i)}\Big)^2\,\mathbb{E}|\Xi^*|^2 = \sum_{i,j=1}^m t_i t_j\,\sigma_{ij}^2\,, \tag{5.3}$$

where $\sigma_{ij}^2 = \lambda\,\big(\mathbb{E}J_0^{(i)}(\Xi^*)J_0^{(j)}(\Xi^*) - \lambda^{(i)}\,\lambda^{(j)}\,\mathbb{E}|\Xi^*|^2\big)$ for $i,j = 1,\ldots,m$. But these quantities coincide with the entries of the matrix K as defined in (3.4). In other words, the ratio $B_\varrho^2(t,X)/|W_\varrho|$ converges a.s. to $\sigma^2(t)$ as $\varrho \to \infty$. This implies the convergence in probability which is just the assertion (5.2).

In this and the subsequent section we use the abbreviation

$$g(t, J_0, C) = t^\top J_0(C) - t^\top \lambda_0\,|C| = \sum_{i=1}^m t_i\,\big(J_0^{(i)}(C) - \lambda^{(i)}\,|C|\big) \tag{5.4}$$

for any $t \in \mathbb{R}^m$ and $C \in \mathcal{C}$, where $\lambda_0 = (\lambda^{(1)}, \ldots, \lambda^{(m)})^\top$.

Lemma 5.2. *Under the condition* (3.2) *of Theorem* 3.1*, for any* $\delta > 0$,

$$\frac{1}{|W_\varrho|}\mathbb{E}\Big(\sum_{n\geq 1}\mathbb{E}_X\big(g^2(t,J_0,\Xi_n\cap W_\varrho)\,\mathbb{I}_{\{\,|\,g(t,J_0,\Xi_n\cap W_\varrho)\,|\,\geq\,\delta\,\sqrt{|W_\varrho|}\}}\big)\Big) \underset{\varrho\to\infty}{\longrightarrow} 0 . \tag{5.5}$$

Proof. Since $g(t,J_0,C) \stackrel{\mathrm{d}}{=} g(t,J_0,C+x)$ for any $C \in \mathcal{C}$ and $x \in \mathbb{R}^d$, by means of the theorems of Campbell and Fubini we get

$$\mathbb{E}\Big(\sum_{n\geq 1} \mathbb{E}_X\big(g^2(t,J_0,\Xi_n \cap W_\varrho)\,\mathbb{1}_{\{\,|\,g(t,J_0,\Xi_n\cap W_\varrho)\,|\,\geq \delta\sqrt{|W_\varrho|}\}}\big)\Big)$$

$$= \lambda \int_{\mathcal{P}^0}\int_{\mathbb{R}^d} \mathbb{E}\Big(g^2(t,J_0,C\cap(W_\varrho - x))\,\mathbb{1}_{\{\,|\,g(t,J_0,C\cap(W_\varrho - x))\,|\,\geq \delta\sqrt{|W_\varrho|}\}}\Big)\,dx\,P^0(dC)\,.$$

By the definition of $g(t,J_0,C)$ introduced in (5.4), it is easy to see that

$$\big|\,g(t,J_0,C\cap(W_\varrho - x))\,\big| \;\leq\; \|t\|\,\Big(\sum_{i=1}^m J_0^{(i)}(C) \;+\; \|\lambda_0\|\,|C|\Big)$$

and that

$$g^2(t,J_0,C\cap(W_\varrho - x)) \;\leq\; 2\|t\|^2\,\Big(\sum_{i=1}^m \big(J_0^{(i)}(C\cap(W_\varrho - x))\big)^2 \;+\; \|\lambda_0\|^2\,|C\cap(W_\varrho - x)|^2\Big)\,.$$

Together with (3.7), this yields the inequality

$$\int_{\mathbb{R}^d} g^2(t,J_0,C\cap(W_\varrho - x))\,dx \;\leq\; 2\|t\|^2\,|W_\varrho|\,\Big(\sum_{i=1}^m \big(J_0^{(i)}(C)\big)^2 \;+\; \|\lambda_0\|^2\,|C|^2\Big)\,.$$

Thus, summarizing the above estimates shows that the left-hand side of (5.5) is bounded by the product of $2\,\lambda\,\|t\|^2$ and

$$\int_{\mathcal{P}^0} \mathbb{E}\Big(\sum_{i=1}^m \Big(\big(J_0^{(i)}(C)\big)^2 \;+\; \|\lambda_0\|^2\,|C|^2\Big)\,\mathbb{1}_{\{\|t\|\,\big(\sum_{i=1}^m J_0^{(i)}(C)+\|\lambda_0\|\,|C|\big)\,\geq\,\delta\sqrt{|W_\varrho|}\}}\Big)\,P^0(dC)\,.$$

By the integrability conditions in (3.2), the latter expression converges to zero as $\varrho \to \infty$.

Finally, in order to prove Theorem 5.1, we need the following generalization of the well–known Berry–Esseen inequality for independent random variables.

Lemma 5.3. *For independent mean zero random variables $U_1\,,U_2\,,\ldots$ with finite variances, there exist absolute constants $a_1,a_2>0$ such that, for any $\varepsilon\in(0,1)$ and $n\in\mathbb{N}$,*

$$\sup_{x\in\mathbb{R}}\Big|\mathbf{P}\Big(\sum_{i=0}^n U_i \leq xB_n\Big) - \Phi(x)\Big| \;\leq\; a_1\,\varepsilon + \frac{a_2}{B_n^2}\sum_{i=1}^n \mathbb{E}U_i^2\,\mathbb{1}_{\{|U_i|\geq\varepsilon B_n\}}\,,$$

where $B_n^2 = \sum_{i=1}^n \mathbb{E}U_i^2$ and Φ denotes the standard normal distribution function on $\mathbb{R}$.

Notice that Lemma 5.3 can be easily obtained from Theorem 5.6 in [20] if in the latter theorem we put $X_i = U_i/B_n$ and consider the function $g:\mathbb{R}\to[0,\infty)$ with

$$g(x) = \begin{cases} \varepsilon & \text{if } |x|<\varepsilon,\\[2ex] |x| & \text{if } \varepsilon\leq|x|<1,\\[2ex] 1 & \text{if } |x|\geq 1.\end{cases}$$

5.2. Proof of Theorem 5.1

Recall that $t^\top \widetilde{Z}_\varrho = \sum_{i=1}^m t_i \widetilde{Z}_\varrho^{(i)}$ and $\sigma^2(t) = t^\top K t$ for $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$. The well–known Cramér-Wold device states that the m–variate central limit theorem (5.1) is equivalent to

$$\lim_{\varrho\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\Big(t^\top \widetilde{Z}_\varrho \le x \Big) - \Phi\Big(\frac{x}{\sigma(t)}\Big) \right| = 0\,, \tag{5.6}$$

for all $t \in \mathbb{R}^m$ with $\sigma^2(t) > 0$, and $t^\top \widetilde{Z}_\varrho \underset{\varrho\to\infty}{\longrightarrow} 0$ in probability if $\sigma^2(t) = 0$. The latter holds since $\mathbb{E}(t^\top \widetilde{Z}_\varrho)^2 \underset{\varrho\to\infty}{\longrightarrow} \sigma^2(t)$ by (3.8). Thus, let $t \in \mathbb{R}^m$ be fixed such that $\sigma^2(t) > 0$. Since the random vector measures $J_n = \big(J_n^{(1)}, \ldots, J_n^{(m)}\big)^\top$ are mutually independent and independent of the tessellation $X = \{\Xi_n\}_{n\ge1}$, we may write

$$\mathbb{P}\Big(t^\top \widetilde{Z}_\varrho \le x\Big) = \mathbb{E}\bigg(\mathbb{P}_X\Big(\sum_{n\ge1} t^\top \big(J_n(\Xi_n \cap W_\varrho) - \lambda_0\, |\Xi_n \cap W_\varrho|\big) \le x\sqrt{|W_\varrho|}\Big)\bigg).$$

Lemma 5.3, with the notation introduced in Section 5.1, yields the estimate

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}_X\Big(\sum_{n\ge1} t^\top \big(J_n(\Xi_n \cap W_\varrho) - \lambda_0\, |\Xi_n \cap W_\varrho|\big) \le x\sqrt{|W_\varrho|}\Big) - \Phi\Big(x\,\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)}\Big) \right|$$
$$\le a_1\varepsilon + \frac{a_2}{B_\varrho^2(t,X)} \sum_{n\ge1} \mathbb{E}_X\Big(g^2(t, J_0, \Xi_n \cap W_\varrho)\, \mathbb{I}_{\{|g(t,J_0,\Xi_n\cap W_\varrho)|\ge\varepsilon\, B_\varrho(t,X)\}}\Big)\,,$$

where $\varepsilon \in (0,1)$ can be chosen arbitrarily small such that $\varepsilon \le \frac{3}{4}\,\sigma^2(t)$. Furthermore, (5.3) and the inequality

$$\left| \frac{B_\varrho^2(t,X)}{|W_\varrho|} - \sigma^2(t) \right| \le \varepsilon \le \frac{3}{4}\,\sigma^2(t)$$

imply that

$$\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} \le \frac{2}{\sigma(t)} \quad \text{and} \quad \left| \frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} - \frac{1}{\sigma(t)} \right| \le \frac{2\,\varepsilon}{\sigma^3(t)}$$

for all $\varrho > 0$ sufficiently large. The mean value theorem, together with $\max_{x\in\mathbb{R}} \Phi'(x) = 1/\sqrt{2\pi}$, yields

$$\left| \Phi\Big(x\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)}\Big) - \Phi\Big(\frac{x}{\sigma(t)}\Big) \right| \le \frac{|x|}{\sqrt{2\pi}} \left| \frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)} - \frac{1}{\sigma(t)} \right|.$$

By the estimates derived above, it is easily seen that

$$\big|\mathbb{P}(t^\top \widetilde{Z}_\varrho \le x) - \Phi(\frac{x}{\sigma(t)})\big| \le \mathbb{P}(E_\varrho^c(t,\varepsilon)) + \mathbb{E}\big(\mathbb{I}_{E_\varrho(t,\varepsilon)} \big|\Phi\big(x\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)}\big) - \Phi(\frac{x}{\sigma(t)})\big|\big)$$
$$+\, \mathbb{E}\big(\mathbb{I}_{E_\varrho(t,\varepsilon)} \big|\mathbb{P}_X\big(\sum_{n\ge1} t^\top \big(J_n(\Xi_n\cap W_\varrho) - \lambda\, |\Xi_n \cap W_\varrho|\big) \le x\sqrt{|W_\varrho|}\big) - \Phi\big(x\frac{\sqrt{|W_\varrho|}}{B_\varrho(t,X)}\big)\big|\big)$$
$$\le \mathbb{P}(E_\varrho^c(t,\varepsilon)) + \frac{2\,\varepsilon\,|x|}{\sqrt{2\pi}\,\sigma^3(t)} + a_1\varepsilon$$
$$+ \frac{4\,a_2}{\sigma^2(t)\,|W_\varrho|}\, \mathbb{E}\bigg(\sum_{n\ge1} \mathbb{E}_X\Big(g^2(t, J_0, \Xi_n \cap W_\varrho)\, \mathbb{I}_{\big\{\big|g(t,J_0,\Xi_n\cap W_\varrho)\big|\ge\varepsilon\,\sigma(t)\,\sqrt{|W_\varrho|}/2\big\}}\Big)\bigg).$$

Since, by Čebychev's inequality, we have

$$\Big|\mathbb{P}\big(t^\top \widetilde{Z}_\varrho \le x\big) \;-\; \Phi\Big(\frac{x}{\sigma(t)}\Big)\Big| \le \varepsilon \,\max\{\mathbb{E}(t^\top \widetilde{Z}_\varrho)^2, \sigma^2(t)\}$$

for $|x| \ge 1/\sqrt{\varepsilon}$, we conclude from (3.8), Lemma 5.1 and Lemma 5.2 that

$$\overline{\lim_{\varrho\to\infty}} \sup_{x\in\mathbf{R}} \Big|\mathbb{P}\Big(t^\top \widetilde{Z}_\varrho \le x\Big) - \Phi\Big(\frac{x}{\sigma(t)}\Big)\Big| \le a_1\varepsilon + \frac{\sqrt{2\,\varepsilon}}{\sqrt{\pi}\,\sigma^3(t)} + \varepsilon\,\sigma^2(t)$$

for any sufficiently small $\varepsilon > 0$. This proves (5.6).

6. Numerical examples

In this section, we assume that $d = 2$ and present some numerical results regarding the asymptotic covariance matrix K as well as the asymptotic distribution of certain functionals. Related numerical results for superpositions of Poisson–Voronoi tessellations can be found, e.g., in [2].

6.1. Poisson nestings

The (initial) tessellation X is chosen to be either a Poisson line tessellation (PLT) or a Poisson-Voronoi tessellation (PVT), where in both cases $\lambda = 0.01$. The inner structure of the cells Ξ_n of X is assumed to be induced by (component) tessellations X_n, where either one of four possible Poisson nesting types $X_n^{(1)}, \ldots, X_n^{(4)}$ is chosen; see Table 1. In each case, X_n can be described by two parameters $\gamma_1 > 0$ and $\gamma_2 > 0$, where

j	1	2	3	4
Type of $X_n^{(j)}$	PLT/PVT	PLT/PLT	PVT/PLT	PVT/PVT

TABLE 1: *Choices of Poisson nesting types for X_n*

γ_2 is the intensity of the nested tessellation. We concentrate on the case $m = 2$, i.e. $J_n = (J_n^{(1)}, J_n^{(2)})^\top$. Particularly, $J_n^{(1)}$ counts the nodes of the component tessellation X_n, and $J_n^{(2)}$ measures the length of the edges of X_n. In order to calculate γ_1 and γ_2, we assume that the vector $(\lambda^{(1)}, \lambda^{(2)})^\top$ is the same for all four choices of X_n. This means that we obtain the system of equations

$$\lambda^{(1)} = g_1^{(j)}(\gamma_1^{(j)}, \gamma_2^{(j)}) \quad\text{and}\quad \lambda^{(2)} = g_2^{(j)}(\gamma_1^{(j)}, \gamma_2^{(j)}) \tag{6.1}$$

for $j = 1, \ldots, 4$. Particularly, the following formulae (6.2) to (6.5) show the system of equations for $X_n^{(1)}, \ldots, X_n^{(4)}$, respectively (see, e.g. [14]):

$$\lambda^{(1)} = 2\gamma_2^{(1)} + \frac{(\gamma_1^{(1)})^2}{\pi} + \frac{8}{\pi}\gamma_1^{(1)}\sqrt{\gamma_2^{(1)}}\,, \qquad \lambda^{(2)} = \gamma_1^{(1)} + 2\sqrt{\gamma_2^{(1)}} \tag{6.2}$$

$$\lambda^{(1)} = \frac{(\gamma_1^{(2)})^2}{\pi} + \frac{(\gamma_2^{(2)})^2}{\pi} + \frac{4}{\pi}\gamma_1^{(2)}\gamma_2^{(2)}\,, \qquad \lambda^{(2)} = \gamma_1^{(2)} + \gamma_2^{(2)} \tag{6.3}$$

$$\lambda^{(1)} = 2\gamma_1^{(3)} + \frac{(\gamma_2^{(3)})^2}{\pi} + \frac{8}{\pi}\gamma_2^{(3)}\sqrt{\gamma_1^{(3)}}\,, \qquad \lambda^{(2)} = \gamma_2^{(3)} + 2\sqrt{\gamma_1^{(3)}} \tag{6.4}$$

$$\lambda^{(1)} = 2\gamma_1^{(4)} + 2\gamma_2^{(4)} + \frac{16}{\pi}\sqrt{\gamma_1^{(4)}\gamma_2^{(4)}}\,, \qquad \lambda^{(2)} = 2\sqrt{\gamma_1^{(4)}} + 2\sqrt{\gamma_2^{(4)}} \tag{6.5}$$

Note that it is difficult, however not impossible, to find values for the intensities $\lambda^{(1)}$ and $\lambda^{(2)}$ such that (6.2) to (6.5) have positive solutions $\gamma_1^{(j)}, \gamma_2^{(j)}$ simultaneously for each $j = 1, \ldots, 4$. As Table 2 shows, we confine ourselves to the case where (6.2) through (6.4) are regarded.

Type of X_n	γ_1	γ_2
PLT/PVT ($X_n^{(1)}$)	0.0857086	0.0000511
PLT/PLT ($X_n^{(2)}$)	0.0848829	0.0151171
PVT/PLT ($X_n^{(3)}$)	0.0000511	0.0857086

TABLE 2: *Intensities of* $X_n^{(1)}, X_n^{(2)}$ *and* $X_n^{(3)}$ *for* $(\lambda^{(1)}, \lambda^{(2)})^\top = (0.004, 0.1)^\top$

In the above examples, the integrability conditions of (3.10) (see also the remark after Theorem 5.1) are obviously satisfied whenever

$$\mathbb{E}D^4(\Xi^*) < \infty \quad \text{and} \quad \mathbb{E}\Big(J_0^{(i)}([0,1)^2)\Big)^2 < \infty\,, \qquad i = 1, 2\,. \tag{6.6}$$

The first part of (6.6) is true because the diameter of the typical cell Ξ^* has an exponentially bounded tail, both for Poisson–Voronoi tessellations and for Poisson line tessellations; see, e.g., [13]. Having in mind that the random measures $J_0^{(i)}$ are induced by the Poisson–type nestings mentioned in Table 1, the second–order properties of Poisson line and Poisson–Voronoi tessellations provide that the second part of (6.6) is satisfied as well.

6.2. Computation of asymptotic covariance matrices

For each of n_1 realizations ξ^* of the typical cell Ξ^* of X, we consider n_2 realizations of $X_n^{(1)}, X_n^{(2)}, X_n^{(3)}$, where the respective intensities are chosen according to Table 2. Hence, a covariance estimator based on the n_2 measured values of $J_n^{(1)}(\xi^*)$ and $J_n^{(2)}(\xi^*)$ can be calculated by using the natural approach $S_{UV} = \frac{1}{n-1}(\sum_{i=1}^n U_i V_i - n\overline{U}\overline{V})$, where $(U_1, \ldots, U_n)^\top$ and $(V_1, \ldots, V_n)^\top$ denote two vectors of sample variables. Finally, the sample mean of the n_1 estimates of covariances is multiplied by λ in order to get an estimate of K. Table 3 shows K for $n_1 = n_2 = 100000$ in the case where X is a PLT. In Table 4, an estimate of K is shown with X being a PVT.

Clearly, the estimated covariance between $J_n^{(1)}(\Xi^*)$ and $J_n^{(2)}(\Xi^*)$ seems to depend strongly on the type of X, while the choice of X_n given a certain X does not yield much difference between the values.

The simulations have been performed using packages of the GeoStoch library [8]; see also [15]. Further simulations evaluating our central limit theorem given in Theorem 5.1 showed that the asymptotic normality of the distribution of $(\widetilde{Z}_\varrho^{(1)}, \widetilde{Z}_\varrho^{(2)})^\top$ is justified quite well, even in the case of a relatively small quadratic sampling window W_ϱ of area $|W_\varrho| = 200^2$.

$$\begin{pmatrix} 0.0122837 & 0.1171656 \\ 0.1171656 & 1.6270537 \end{pmatrix} \quad \begin{pmatrix} 0.0128533 & 0.1220669 \\ 0.1220669 & 1.6841176 \end{pmatrix} \quad \begin{pmatrix} 0.0125047 & 0.1137313 \\ 0.1137313 & 1.6069373 \end{pmatrix}$$

(a) *Nesting type* $X_n^{(1)}$ (b) *Nesting type* $X_n^{(2)}$ (c) *Nesting type* $X_n^{(3)}$

TABLE 3: *Asymptotic covariance matrix K if X is a PLT with* $\lambda = 0.01$

$$\begin{pmatrix} 0.0090550 & 0.0711104 \\ 0.0711104 & 0.9873148 \end{pmatrix} \quad \begin{pmatrix} 0.0090969 & 0.0711946 \\ 0.0711946 & 0.9913954 \end{pmatrix} \quad \begin{pmatrix} 0.0093042 & 0.0701055 \\ 0.0701055 & 0.9832765 \end{pmatrix}$$

(a) *Nesting type* $X_n^{(1)}$ (b) *Nesting type* $X_n^{(2)}$ (c) *Nesting type* $X_n^{(3)}$

TABLE 4: *Asymptotic covariance matrix K if X is a PVT with* $\lambda = 0.01$

7. Conclusion

In this paper, a (normalized) vector of functionals $\widetilde{Z}_\varrho$ has been considered, where its components describe the inner structure of the cells of a stationary and ergodic tessellation X. We have shown that under certain conditions the distribution function of $\widetilde{Z}_\varrho$ converges uniformly to the (m-dimensional) multivariate normal distribution $\mathrm{N}(o, K)$ if the sampling window W_ϱ grows unboundedly as $\varrho \to \infty$. The asymptotic covariance matrix K has been determined and laws of large numbers are shown, which provide unbiased and consistent estimators for the intensities $\lambda^{(1)}, \ldots, \lambda^{(m)}$ of the stationary random measures $J_0^{(1)}, \ldots, J_0^{(m)}$.

There are several interesting perspectives for further research. In particular, the vector of functionals $J_n^{(1)}, \ldots, J_n^{(m)}$ can be generalized such that each component $J_n^{(i)}$ of this vector is a functional defined on the k-facets $(0 \le k \le d)$ of the cell Ξ_n of X. For example, if $d = 2$ and $J_n^{(i)}$ is induced by the stationary tessellation X_n, then $\top$–crossings can be analyzed on the boundary of Ξ_n, induced by the edges of a nested tessellation X_n .

Another interesting problem is the derivation of an unbiased and consistent estimator for the asymptotic covariance matrix $K = \lim_{\varrho\to\infty} \mathsf{Cov}\widetilde{Z}_\varrho$. Such an estimator is needed if one wants to construct asymptotic hypothesis tests for the intensities $\lambda^{(1)}, \ldots, \lambda^{(m)}$ of $J_0^{(1)}, \ldots, J_0^{(m)}$, based on the central limit theorem stated in Theorem 5.1.

Acknowledgements

This research was supported by France Télécom through research grant 42 36 68 97.

References

[1] F. Avram and D. Bertsimas (1993). On central limit theorems in geometrical probability. *Annals of Applied Probability* **3,** 1033–1046.

[2] F. Baccelli, C. Gloaguen and S. Zuyev (2000). Superposition of planar Voronoi tessellations. *Communications in Statistics, Ser. Stochastic Models* **16,** 69–98.

[3] A.J. BADDELEY (1980). A limit theorem for statistics of spatial data. *Advances in Applied Probability* **12,** 447–461.

[4] S. BÖHM, L. HEINRICH AND V. SCHMIDT (2004). Asymptotic properties of estimators for the volume fractions of jointly stationary random sets. *Statistica Neerlandica* **58,** 1–19.

[5] R. COWAN (1980). Properties of ergodic random mosaic processes. *Mathematische Nachrichten* **97,** 89–102.

[6] D.J. DALEY AND D. VERE-JONES (1988). *An Introduction to the Theory of Point Processes.* Springer, New York.

[7] C. DELLACHERIE AND P.-A. MEYER (1994). *Probabilities and Potential.* North–Holland, Amsterdam.

[8] GEOSTOCH (2004). Java library. University of Ulm, Department of Applied Information Processing and Department of Stochastics (http://www.geostoch.de).

[9] C. GLOAGUEN, P. COUPÉ, R. MAIER, AND V. SCHMIDT (2002). Stochastic modelling of urban access networks. *Proceedings of IEEE Infocom '02*, München.

[10] L. HEINRICH (1994). Normal approximation for some mean-value estimates of absolutely regular tessellations. *Mathematical Methods of Statistics* **3,** 1–24.

[11] L. HEINRICH AND I.S. MOLCHANOV (1999). Central limit theorem for a class of random measures associated with germ-grain models. *Advances in Applied Probability* **31,** 283–314.

[12] K. LEICHTWEISS (1980). *Konvexe Mengen.* Springer, Berlin.

[13] R. MAIER, J. MAYER AND V. SCHMIDT (2004). Distributional properties of the typical cell of stationary iterated tessellations. *Mathematical Methods of Operations Research* **59,** 287–302.

[14] R. MAIER AND V. SCHMIDT (2003). Stationary iterated tessellations. *Advances in Applied Probability* **35,** 337–353.

[15] J. MAYER, V. SCHMIDT AND F. SCHWEIGGERT (2004). A unified simulation framework for spatial stochastic models. *Simulation Modelling Practice and Theory* **12**, 307–326.

[16] S. MASE (1982). Asymptotic properties of stereological estimators of the volume fraction for stationary random sets. *Journal of Applied Probability* **19,** 111–126.

[17] J. MØLLER (1989). Random tessellations in $\mathbb{R}^d$. *Advances in Applied Probability* **21,** 37–73.

[18] A. OKABE, B. BOOTS, K. SUGIHARA AND S.N. CHIU (2000). *Spatial Tessellations.* Second Edition. J.Wiley & Sons, Chichester.

[19] K. PAROUX (1998). Quelques théorèmes centraux limites pour les Processus poissoniens de droites dans le plan. *Advances in Applied Probability* **30,** 640–656.

[20] V.V. PETROV (1995). *Limit Theorems of Probability Theory. Sequences of Independent Random Variables.* Clarendon Press, Oxford.

[21] V. SCHMIDT AND E. SPODAREV (2004). Joint estimators for the specific intrinsic volumes of stationary random sets. *Stochastic Processes and Their Applications* (to appear).

[22] R. SCHNEIDER AND W. WEIL (2000). *Stochastische Geometrie.* Teubner, Stuttgart.

[23] D. STOYAN, W.S. KENDALL AND J. MECKE (1995). *Stochastic Geometry and its Applications.* Second Edition. J. Wiley & Sons, Chichester.